

# The No-Pole Condition in Landau gauge: Properties of the Gribov Ghost Form-Factor and a Constraint on the $2d$ Gluon Propagator

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We study general properties of the Landau-gauge Gribov ghost form-factor  $\sigma(p^2)$  for  $SU(N_c)$  Yang-Mills theories in the  $d$ -dimensional case. We find a qualitatively different behavior for  $d = 3, 4$  with respect to the  $d = 2$  case. In particular, considering any (sufficiently regular) gluon propagator  $\mathcal{D}(p^2)$  and the one-loop-corrected ghost propagator, we prove in the  $2d$  case that the function  $\sigma(p^2)$  blows up in the infrared limit  $p \rightarrow 0$  as  $-\mathcal{D}(0) \ln(p^2)$ . Thus, for  $d = 2$ , the no-pole condition  $\sigma(p^2) < 1$  (for  $p^2 > 0$ ) can be satisfied only if the gluon propagator vanishes at zero momentum, that is,  $\mathcal{D}(0) = 0$ . On the contrary, in  $d = 3$  and  $4$ ,  $\sigma(p^2)$  is finite also if  $\mathcal{D}(0) > 0$ . The same results are obtained by evaluating the ghost propagator  $\mathcal{G}(p^2)$  explicitly at one loop, using fitting forms for  $\mathcal{D}(p^2)$  that describe well the numerical data of the gluon propagator in two, three and four space-time dimensions in the  $SU(2)$  case. These evaluations also show that, if one considers the coupling constant  $g^2$  as a free parameter, the ghost propagator admits a one-parameter family of behaviors (labelled by  $g^2$ ), in agreement with previous works by Boucaud et al. In this case the condition  $\sigma(0) \leq 1$  implies  $g^2 \leq g_c^2$ , where  $g_c^2$  is a “critical” value. Moreover, a free-like ghost propagator in the infrared limit is obtained for any value of  $g^2$  smaller than  $g_c^2$ , while for  $g^2 = g_c^2$  one finds an infrared-enhanced ghost propagator. Finally, we analyze the Dyson-Schwinger equation for  $\sigma(p^2)$  and show that, for infrared-finite ghost-gluon vertices, one can bound the ghost form-factor  $\sigma(p^2)$ . Using these bounds we find again that only in the  $d = 2$  case does one need to impose  $\mathcal{D}(0) = 0$  in order to satisfy the no-pole condition. The  $d = 2$  result is also supported by an analysis of the Dyson-Schwinger equation using a spectral representation for the ghost propagator. Thus, if the no-pole condition is imposed, solving the  $d = 2$  Dyson-Schwinger equations cannot lead to a massive behavior for the gluon propagator. These results apply to any Gribov copy inside the so-called first Gribov horizon, i.e. the  $2d$  result  $\mathcal{D}(0) = 0$  is not affected by Gribov noise. These findings are also in agreement with lattice data.

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## I. INTRODUCTION

Green functions of Yang-Mills theories are gauge-dependent quantities. They can, however, be used as a starting point for the evaluation of hadronic observables (see for example [1–4]). Thus, the study of the infrared (IR) behavior of propagators and vertices is an important step in our understanding of QCD. In particular, the confinement mechanism for color charges [5] could reveal itself in the IR behavior of (some of) these Green functions. This IR behavior should also be relevant for the description of the deconfinement transition and of the deconfined phase of QCD. Indeed, at high temperature color charges are expected to be Debye-screened and the (electric and magnetic) screening masses should be related to the IR behavior of the gluon propagator (see for example [6–8]).

Among the gauge-fixing conditions employed in studies of Yang-Mills Green functions, a very popular choice is the Landau gauge, which in momentum space reads  $p_\mu A_\mu^b(p) = 0$ . From the continuum perspective this gauge has various important properties, including its renormalizability, various associated nonrenormalization theorems [9] and a ghost-antighost symmetry [1]. In the past few years many analytic studies of Green functions in Landau gauge have focused on the solution of the Yang-Mills Dyson-Schwinger equations (DSEs), which are the exact quantum equations of motion of the theory (see for example [1, 2, 10, 11]). Since the DSEs are an infinite set of coupled equations, any attempt of solving them requires a truncation scheme. Then, some Green functions (usually the gluon and the ghost propagators) are obtained self-consistently from the considered equations, while all the other Green functions entering the equations are given as an input. For the coupled DSEs of gluon and ghost propagators two solutions have been extensively analyzed (see for example Chapter 10 in Ref. [5] and Ref. [12] for recent short reviews). The scaling solution [1, 13–16] finds in  $d = 2, 3$  and 4 an IR-enhanced ghost propagator  $\mathcal{G}(p^2)$  and a vanishing gluon propagator  $\mathcal{D}(p^2)$  at zero momentum.<sup>1</sup> In particular, the IR behavior of the two propagators should be given respectively by  $\mathcal{G}(p^2) \sim (p^2)^{-\kappa_G-1}$  and by  $\mathcal{D}(p^2) \sim (p^2)^{2\kappa_D+(2-d)/2}$  with  $\kappa_G = \kappa_D \approx 0.2(d-1)$  [14, 16]. On the other hand, the massive solution [18–26] gives (for  $d = 3$  and 4) a free-like ghost propagator in the IR limit, i.e.  $\kappa_G = 0$ , and a massive behavior for the gluon propagator,<sup>2</sup> that is,  $\mathcal{D}(0) > 0$  and  $\kappa_D = (d-2)/4$ .

The existence of two different types of solution for the coupled gluon and ghost DSEs is now understood as due to the use of different auxiliary *boundary conditions*.<sup>3</sup> These conditions can be given in terms of the value of the ghost dressing function  $\mathcal{F}(p^2) = p^2 \mathcal{G}(p^2)$  at a given momentum scale  $p$  [22, 31]. In particular, if one considers  $p = 0$ , it is clear that  $1/\mathcal{F}(0) = 0$  gives an IR-enhanced ghost propagator  $\mathcal{G}(p^2)$  while  $1/\mathcal{F}(0) > 0$  yields a free-like behavior for  $\mathcal{G}(p^2)$  at small momenta. As stressed in Ref. [22] (see also the discussion in Section 4.2.2 of Ref. [12]), the scaling condition  $1/\mathcal{F}(0) = 0$  relies on a particular cancellation in the ghost DSE which, in turn, implies a specific “critical” value  $g_c^2$  for the coupling constant  $g^2$  [12, 32]. Thus, at least from the mathematical point of view, there is a one-parameter family of solutions for the gluon and ghost DSEs, labelled by  $g^2$  or, equivalently, by  $1/\mathcal{F}(0)$ : in the case  $g^2 = g_c^2$  one recovers the scaling solution while, for all cases  $g^2 < g_c^2$ , the solution is a massive one.<sup>4</sup> In  $4d$ , the  $SU(3)$  physical value of the coupling seems to select<sup>5</sup> the massive solution [21, 34].

At this point we should recall that, when considering gauge-dependent quantities in non-Abelian gauge theories, one has to deal with the existence of Gribov copies [35] (see also Ref. [36] for a recent review). Indeed, for compact non-Abelian Lie groups defined on the 4-sphere [37] or on the 4-torus [38], it is impossible to find a continuous choice of one (and only one) connection  $A_\mu(x)$  on each gauge orbit. The effect of Gribov copies is not seen in perturbation theory [35], i.e. the usual Faddeev-Popov-quantization procedure is correct at the perturbative level. However, these copies could be relevant at the non-perturbative level, i.e. in studies of the IR properties of Yang-Mills theories.

Different approaches have been proposed in order to quantize a Yang-Mills theory while taking into account the existence of Gribov copies (see for example [35, 39–45]). The one usually considered, both in the continuum and on the

<sup>1</sup> For the explanation of color confinement based on the scaling solution see for example Section 3.4 in [17] and references therein.

<sup>2</sup> The possible existence of a dynamical mass for the gluons, as well as its relation to quark confinement through vortex condensation, has been discussed a long time ago in Ref. [27].

<sup>3</sup> The possibility of having different non-perturbative solutions for DSEs in relation with different boundary conditions has been discussed for example in Refs. [28, 29] (see also Section 1.4 in [30] and Sections 3 and 3.1 in Ref. [1]).

<sup>4</sup> If  $g^2 > g_c^2$  one gets a negative ghost propagator at small momenta.

<sup>5</sup> Recently, in Ref. [33], it has also been shown — using a renormalization-group approach — that in three and four space-time dimensions only the decoupling solution is expected to be physically realized.

lattice, is based on restricting the functional integration to a subspace of the hyperplane of transverse configurations. The original proposal, made by Gribov [35], was based on the observation that the Landau gauge condition  $\partial_\mu A_\mu^b(x) = 0$  allows for (infinitesimally) gauge-equivalent configurations if the Landau-gauge Faddeev-Popov operator  $\mathcal{M}^{bc}(x, y) = -\delta(x - y)\partial_\mu D_\mu^{bc}$  has zero modes. (Here  $D_\mu^{bc}$  is the usual covariant derivative.) Indeed, since an infinitesimal gauge transformation  $\delta\omega(x)$  gives  $A_\mu^b(x) \rightarrow A_\mu^b(x) + D_\mu^{bc}\omega^c(x)$ , it is clear that the exclusion (in the path-integral measure) of the zero modes of  $\mathcal{M}^{bc}(x, y)$  implies that gauge copies connected by such infinitesimal gauge transformations are ignored in the computation of expectation values. In order to exclude these zero modes, Gribov considered a stronger condition by requiring that the functional integration be restricted to the region  $\Omega$  of gauge configurations  $A_\mu^b(x)$  defined as

$$\Omega \equiv \{ A_\mu^b(x) : \partial_\mu A_\mu^b(x) = 0, \mathcal{M}^{bc}(x, y) > 0 \} . \quad (1)$$

This set, known as the (first) Gribov region, clearly includes the vacuum configuration  $A_\mu^b(x) = 0$ , for which the Faddeev-Popov operator is given by  $-\delta(x - y)\delta^{bc}\partial_\mu^2$ . The region  $\Omega$  can also be defined (see for example [46, 47]) as the whole set of local minima<sup>6</sup> of the functional  $\mathcal{E}[A] = \int d^d x A_\mu^b(x) A_\mu^b(x)$ . Since usually each orbit allows for more than one local minimum of  $\mathcal{E}[A]$ , it is clear that the region  $\Omega$  is not free of Gribov copies. On the contrary, in the interior of the so-called fundamental modular region  $\Lambda$ , given by the set of the absolute minima of the functional  $\mathcal{E}[A]$ , no Gribov copies occur [43, 48].

The characterization of the fundamental modular region  $\Lambda$ , i.e. finding the absolute minima of the *energy* functional  $\mathcal{E}[A]$ , is a problem similar to the determination of the ground state of a spin glass system [49, 50]. Thus, a local formulation of a Yang-Mills theory, with the functional measure delimited to  $\Lambda$ , is not available, whereas a practical way of restricting the physical configuration space to the region  $\Omega$  was introduced by Gribov [35]. To this end, he required that the ghost dressing function  $\mathcal{F}(p^2)$  cannot have a pole at finite nonzero momenta. After setting

$$\mathcal{G}(p^2) = \frac{\mathcal{F}(p^2)}{p^2} = \frac{1}{p^2} \frac{1}{1 - \sigma(p^2)} , \quad (2)$$

this condition can be written as

$$\sigma(p^2) < 1 \quad \text{for} \quad p^2 > 0 , \quad (3)$$

where  $\sigma(p^2)$  is the so-called Gribov ghost form-factor [35]. Indeed, since the ghost propagator is given by

$$\mathcal{G}(p^2) = \frac{\delta^{bc}}{N_c^2 - 1} \left\langle p \left| (\mathcal{M}^{-1})^{bc} \right| p \right\rangle , \quad (4)$$

i.e. it is related to the inverse of the Faddeev-Popov matrix  $\mathcal{M}^{bc}(x, y)$ , the above inequality — known as the no-pole condition — should be equivalent to the restriction of the functional integration to the Gribov region  $\Omega$ , i.e. to the condition  $\mathcal{M}^{bc}(x, y) > 0$ .

From the discussion above, it is clear that both scaling and massive solutions of DSEs satisfy the no-pole condition, i.e.  $1/\mathcal{F}(p^2) = 1 - \sigma(p^2) > 0$  for  $p^2 > 0$ . Indeed, in the scaling case [1, 13–16], this condition [together with the condition  $1/\mathcal{F}(0) = 0$ ] is imposed from the beginning to the solution of the DSEs. On the contrary, for the massive solution, the no-pole condition is either verified a posteriori, as in Ref. [20], or used [together with the condition  $1/\mathcal{F}(0) > 0$ ] as an input for the solution of the DSEs, as in Refs. [19, 25, 26]. In particular, in Ref. [26], the value of  $1/\mathcal{F}(0)$  is fixed using lattice data.

The restriction to the first Gribov region  $\Omega$  is also always implemented in lattice numerical simulations of Green functions in Landau gauge by (numerically) finding local minima of the functional  $\mathcal{E}[A]$ . Results obtained using very large lattice volumes [51–54] (see also Chapter 10 in Ref. [5], Section 3 in Ref. [12] and Ref. [55] for recent short

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<sup>6</sup> For this reason this gauge condition is often indicated in the literature as *minimal Landau gauge*.

reviews) have shown that in  $d = 3$  and  $4$  the gluon propagator  $\mathcal{D}(p^2)$  is finite and nonzero in the limit  $p \rightarrow 0$  while the ghost propagator  $\mathcal{G}(p^2)$  behaves as  $1/p^2$ . On the contrary, for  $d = 2$  the lattice data [56–59] are in quantitative agreement with the scaling solution and one finds  $\kappa_D = \kappa_G \approx 0.2$ .

Since the region  $\Omega$  is not free of Gribov copies, their (possible) influence on the numerical evaluation of gluon and ghost propagators has been studied by various groups [61–66]. It has been found that these effects are usually observable only in the IR limit and that any attempt to restrict the functional integration to the fundamental modular region  $\Lambda$  gives a stronger suppression at small momenta for both propagators, i.e. reducing the value of  $\mathcal{D}(0)$  and increasing that of  $1/\mathcal{F}(0)$ . More recently, it has been suggested [67–70] that the one-parameter family of solutions obtained for the gluon and ghost DSEs should be related<sup>7</sup> to Gribov-copy effects and that the value of  $1/\mathcal{F}(0)$  could be used as a gauge-fixing parameter. This analysis finds indeed IR-enhanced ghost propagators (and sometimes a disconcerting over-scaling<sup>8</sup>). On the other hand, the gluon propagator still shows a finite nonzero value at zero momentum, that is,  $\mathcal{D}(0) > 0$ . Moreover, this approach does not explain why the numerical results found in  $d = 2$  are different from those obtained in  $d = 3$  and  $4$ , even though Gribov copies inside the first Gribov region  $\Omega$  are clearly present in any space-time dimension  $d > 1$ .

From the analytical point of view, following Gribov’s approach, Zwanziger modified the usual Yang-Mills action in order to restrict the path integral to the first Gribov region  $\Omega$  [71]. Although this restriction is obtained using a non-local term, the Gribov-Zwanziger (GZ) action<sup>9</sup> can be written as a local action and it is proven [72–74] to be renormalizable. At tree level the GZ gluon propagator is given by  $\mathcal{D}(p^2) = p^2/(p^4 + \lambda^4)$ , where  $\lambda$  is a parameter with mass-dimension 1. At the same time, the ghost propagator is given by  $\mathcal{G}(p^2) \sim 1/p^4$ . Thus, as in the scaling solution of the gluon and ghost DSEs, the gluon propagator is null at zero momentum<sup>10</sup> and the ghost propagator is IR-enhanced [42]. These tree-level results, also in agreement with the original work by Gribov [35], have been confirmed by one-loop calculations in the three- and four-dimensional cases [77–81].

More recently, the GZ action has been modified by considering (for  $d = 3$  and  $4$ ) dimension-two condensates [82–85]. The corresponding action, called the Refined Gribov-Zwanziger (RGZ) action, still imposes the restriction of the functional integration to the region  $\Omega$  and it is renormalizable. However, the RGZ action allows for a finite nonzero value of  $\mathcal{D}(0)$  and for a free-like ghost propagator  $\mathcal{G}(p^2)$  in the IR limit. Thus, nonzero values for these dimension-two condensates yield for the gluon and ghost propagators an IR behavior in agreement with the massive solution of the gluon and ghost DSEs.<sup>11</sup> Indeed, the RGZ tree-level gluon propagator describes well the numerical data in the SU(2) case [59, 60], for  $d = 3$  and  $4$ , and in the SU(3) case [88] with  $d = 4$ . It is also interesting to note that the fitting values for the dimension-two condensates are very similar for the SU(2) and SU(3) gauge groups in the four-dimensional case.

As stressed above, the restriction of the functional integration to the first Gribov region  $\Omega$  and the no-pole condition (3) are key ingredients in the study of the IR sector of Yang-Mills theories in Landau gauge. However, to our knowledge, a detailed investigation of the properties of the Gribov form-factor  $\sigma(p^2)$  as well as of the possible implications of the no-pole condition was missing up to now, although some interesting one-loop results were already presented in Refs. [84, 89–91]. In particular, in Appendix B.2 of [84] it was shown that, if the gluon propagator  $\mathcal{D}(p^2)$  is positive, then in the  $2d$  case the derivative  $\partial\sigma(p^2)/\partial p^2$  is negative for all values of  $p^2$ , i.e.  $\sigma(p^2)$  is largest at  $p^2 = 0$ . Also, in Ref. [89] it was proven that in the RGZ framework the form-factor  $\sigma(p^2)$  presents a logarithmic IR singularity  $-\ln(p^2)$  for  $d = 2$ . This result precluded the use of the RGZ action in the two-dimensional case, leading to a first interpretation of the different behavior found in lattice numerical simulations for the  $2d$  case, compared to the  $d = 3$  and  $4$  cases. Similar findings have been (more recently) presented in Refs. [33, 92, 93].

In this work we collect some general properties of the Gribov form-factor  $\sigma(p^2)$  and we study the consequences of imposing the no-pole condition. In particular, in Section II, using the expression for  $\sigma(p^2)$  obtained from the

<sup>7</sup> This identification is, however, based on several (unproven) hypotheses, as already stressed in Ref. [58].

<sup>8</sup> Let us recall that the scaling solution is supposed to be unique [31].

<sup>9</sup> See again Ref. [36] for a comprehensive review of the GZ action.

<sup>10</sup> One can, however, obtain a finite nonzero value for  $\mathcal{D}(0)$  within the GZ approach by considering a non-analytic behavior for the free energy of the system [75, 76].

<sup>11</sup> Let us note here that a massive behavior for these two propagators has also been obtained in Refs. [86, 87] using different analytic approaches.



FIG. 1. Feynman diagrams for the one-loop-corrected Landau-gauge ghost propagator. Dashed lines represent ghosts, the curly line represents gluons.

evaluation of the ghost propagator at one loop, we prove that  $\sigma(p^2)$  attains its maximum value at  $p^2 = 0$  for any dimension  $d \geq 2$ . Since this expression for  $\sigma(p^2)$  depends on the gluon propagator  $\mathcal{D}(p^2)$ , in the same section we also investigate (for a general  $d$ -dimensional space-time) which IR behavior of the gluon propagator is necessary in order to satisfy the no-pole condition  $\sigma(p^2) < 1$ . By considering a generic (and sufficiently regular) gluon propagator  $\mathcal{D}(p^2)$ , we find in the  $d = 2$  case that  $\sigma(p^2)$  is unbounded unless the gluon propagator is null at zero momentum. More exactly, we find  $\sigma(p^2) \sim -\mathcal{D}(0) \ln(p^2)$  in the  $p \rightarrow 0$  limit, in agreement with [89]. This result does not apply to the  $d = 3$  and 4 cases. Indeed, in these cases one can introduce, for all values of  $p^2$ , simple finite upper bounds for the Gribov form-factor. In Section III we present explicit one-loop calculations for  $\sigma(p^2)$  using for the gluon propagator  $\mathcal{D}(p^2)$  linear combinations of Yukawa-like propagators (with real and/or complex-conjugate poles), which have been recently used to model lattice data of the gluon propagator in the SU(2) case [59, 60]. Besides confirming the results obtained in Section II, we also find that the ghost propagator admits a one-parameter family of behaviors [21] labelled by the coupling constant  $g^2$ , considered as a free parameter. Moreover, the massive solution  $\mathcal{G}(p^2) \sim 1/p^2$ , corresponding to  $\sigma(0) < 1$ , is obtained for all values of  $g^2$  smaller than a “critical” value  $g_c^2$ . At the “critical” value  $g_c^2$ , implying  $\sigma(0) = 1$ , one finds an IR-enhanced ghost propagator. (As already stressed above, the case  $g^2 > g_c^2$  corresponds to  $\sigma(0) > 1$  and one obtains a negative ghost propagator at small momenta.) Finally, in Section IV, we analyze the DSE for  $\sigma(p^2)$ . We stress that in this case we do not try to solve the DSE but we focus only on general properties of this equation. As we will see, considering IR-finite ghost-gluon vertices, we confirm and extend the one-loop analysis of the no-pole condition presented in Section II. In particular, after introducing bounds for the Gribov form-factor, we show again for  $d = 2$  that the gluon propagator  $\mathcal{D}(p^2)$  must vanish at zero momentum in order to keep  $\sigma(p^2)$  finite. On the contrary, such a constraint does not apply in the three- and four-dimensional cases. We also present alternative evidence for the  $d = 2$  result using a spectral representation for the ghost propagator in the DSE.

It is important to note that all our results in Sections II and IV apply irrespective of which set of Gribov copies (inside the region  $\Omega$ ) is considered, i.e. they are not affected by the so-called Gribov noise. We end with our Conclusion in Section V. Some technical details have been collected in four Appendices. In particular, in Appendix B we present properties of the Gauss hypergeometric function  ${}_2F_1(a, b; c; z)$  that are relevant to prove some of our results.

## II. THE ONE-LOOP-CORRECTED GHOST PROPAGATOR AND THE GRIBOV FORM-FACTOR

In this Section, as well as in Section III below, we consider the one-loop-corrected Landau-gauge ghost propagator, diagrammatically represented in Figure 1. This propagator can be written [for the SU( $N_c$ ) gauge group in the  $d$ -dimensional case] as

$$\mathcal{G}(p^2) = \frac{1}{p^2} - \frac{\delta^{ab}}{N_c^2 - 1} \frac{1}{p^4} g^2 f^{adc} f^{cdb} \int \frac{d^d q}{(2\pi)^d} (p - q)_\mu p_\nu \mathcal{D}(q^2) P_{\mu\nu}(q) \frac{1}{(p - q)^2}, \quad (5)$$

where  $\delta^{ab} \mathcal{D}(q^2) \mathcal{P}_{\mu\nu}(q)$  is the tree-level gluon propagator [not necessarily given by  $\mathcal{D}(q) = 1/q^2$ ] and  $\mathcal{P}_{\mu\nu}(q) = (\delta_{\mu\nu} - q_\mu q_\nu / q^2)$  is the usual projector onto the transverse sub-space, i.e.  $q_\mu \mathcal{P}_{\mu\nu}(q) = 0$ . We have also considered the tree-level ghost-gluon vertex  $ig f^{adc} p_\nu$ , where  $p$  is the outgoing ghost momentum. The indices  $a, d, c$  refer, respectively, to the incoming ghost, to the gluon and to the outgoing ghost. After using  $f^{adc} f^{cdb} = -N_c \delta^{ab}$ , valid for the adjoint representation, we obtain

$$\mathcal{G}(p^2) = \frac{1}{p^2} [1 + \sigma(p^2)], \quad (6)$$

where  $\sigma(p^2)$  is the momentum-dependent function

$$\sigma(p^2) = g^2 N_c \frac{p_\mu p_\nu}{p^2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(p-q)^2} \mathcal{D}(q^2) \mathcal{P}_{\mu\nu}(q). \quad (7)$$

Finally, we can write [as in Eq. (2)]

$$\mathcal{G}(p^2) = \frac{1}{p^2} \frac{1}{1 - \sigma(p^2)}, \quad (8)$$

which corresponds to the usual resummation of an infinite set of diagrams into the self-energy. Note that this resummation makes sense only when  $\sigma(k^2) < 1$ , i.e. when the no-pole condition (3) is satisfied.

Clearly the function  $\sigma(p^2)$  is dimensionless and it should go to zero for  $p \rightarrow \infty$ , modulo possible logarithmic corrections. Also, this function coincides with the so-called Gribov ghost form-factor [35, 90, 91], even though the latter is obtained in a slightly different way.<sup>12</sup> As discussed in the Introduction, the no-pole condition  $\sigma(p^2) < 1$  for  $p^2 > 0$  should be equivalent to the restriction of the path integral to the first Gribov region  $\Omega$  [defined in Eq. (1)]. In this section we will derive general properties of  $\sigma(p^2)$  in  $d \geq 2$  space-time dimensions. In particular, as we will see below, the weaker condition  $\sigma(p^2) < +\infty$  is already sufficient to obtain a strong constraint on the IR behavior of the gluon propagator  $\mathcal{D}(p^2)$  in the  $d = 2$  case.

### A. First Derivative of $\sigma(p^2)$ for $d = 2$

Following Appendix B.2 of Ref. [84] one can show that, if the gluon propagator  $\mathcal{D}(q^2)$  is positive, then in the  $2d$  case and for all values of  $p^2$  we have

$$\frac{\partial \sigma(p^2)}{\partial p^2} < 0, \quad (9)$$

with  $\sigma(p^2)$  defined in Eq. (7). This implies that  $\sigma(p^2)$  attains its maximum value at  $p^2 = 0$ . To this end, we choose the positive  $x$  direction parallel to the external momentum  $p$  and write (using polar coordinates)

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{p_\mu p_\nu}{p^2} \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(p-q)^2} \mathcal{D}(q^2) \mathcal{P}_{\mu\nu}(q) = \int_0^\infty \frac{q dq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{p^2 + q^2 - 2qp \cos(\theta)}. \quad (10)$$

The integral in  $d\theta$  can be evaluated using contour integration on the unit circle and the residue theorem. This yields

$$\int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{p^2 + q^2 - 2qp \cos(\theta)} = \oint \frac{i dz}{4pq} \frac{2 - z^2 - \bar{z}^2}{z^2 - z(p/q + q/p) + 1} = \frac{\pi}{p^2} \theta(p^2 - q^2) + \frac{\pi}{q^2} \theta(q^2 - p^2), \quad (11)$$

where  $\oint dz$  represents the integral on the unit circle  $|z| = 1$ , we indicated with  $\bar{z}$  the complex-conjugate of  $z = e^{i\theta}$  and  $\theta(x)$  is the step function. This integral is also evaluated in Eqs. (A17) and (A18) in the Appendix A (for the general  $d$ -dimensional case). Considering also Eq. (B8) and Eq. (A5), with  $d = 2$ , we find

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{4\pi} \left[ \int_0^p \frac{dq}{p^2} q \mathcal{D}(q^2) + \int_p^\infty \frac{dq}{q} \mathcal{D}(q^2) \right] \quad (12)$$

$$= \frac{1}{8\pi} \left[ \int_0^{p^2} \frac{dq^2}{p^2} \mathcal{D}(q^2) + \int_{p^2}^\infty \frac{dq^2}{q^2} \mathcal{D}(q^2) \right] \quad (13)$$

$$= \frac{1}{8\pi} \int_0^\infty dq^2 \mathcal{D}(q^2) \left[ \frac{\theta(p^2 - q^2)}{p^2} + \frac{\theta(q^2 - p^2)}{q^2} \right]. \quad (14)$$

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<sup>12</sup> Note that in the Gribov ghost form-factor there is usually an extra factor  $1/(N_c^2 - 1)$  [90, 91] compared to our Eq. (7). However, this is due to the fact that in Eq. (5) above we considered for the Landau-gauge gluon propagator the usual expression  $D_{\mu\nu}^{ab}(q^2) = \delta^{ab} \mathcal{P}_{\mu\nu}(q) \mathcal{D}(q^2)$  while, in the derivation of the Gribov ghost form-factor, one usually writes  $A_\mu^a(q) A_\nu^a(-q) = \omega(q^2) \mathcal{P}_{\mu\nu}(q)$ , as in Eq. (255) of Ref. [90].

Then, by using  $\partial_x \theta(x) = \delta(x)$ , where  $\delta(x)$  is the Dirac delta function, the derivative of  $\sigma(p^2)$  with respect to  $p^2$  yields

$$\frac{\partial \sigma(p^2)}{\partial p^2} = -\frac{g^2 N_c}{8\pi} \int_0^\infty dq^2 \mathcal{D}(q^2) \frac{\theta(p^2 - q^2)}{p^4} = -\frac{g^2 N_c}{8\pi p^4} \int_0^{p^2} dq^2 \mathcal{D}(q^2), \quad (15)$$

which is clearly negative, for any value of  $p^2$ , if  $\mathcal{D}(q^2)$  is positive. We can evaluate the limit  $p^2 \rightarrow 0$  of this derivative using, for example, the trapezoidal rule<sup>13</sup>

$$\lim_{p^2 \rightarrow 0} \frac{\partial \sigma(p^2)}{\partial p^2} = -\lim_{p^2 \rightarrow 0} \frac{g^2 N_c}{8\pi p^4} \frac{p^2}{2} [\mathcal{D}(p^2) + \mathcal{D}(0)] = -\lim_{p^2 \rightarrow 0} \frac{g^2 N_c \mathcal{D}(0)}{8\pi p^2}. \quad (17)$$

One arrives at the same result after writing Eq. (15) as

$$\frac{\partial \sigma(p^2)}{\partial p^2} = -\frac{g^2 N_c}{8\pi p^2} \int_0^1 dx \mathcal{D}(xp^2) = -\frac{g^2 N_c}{8\pi p^2} \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2}, \quad (18)$$

where  $\hat{D}(p^2)$  is a primitive of  $\mathcal{D}(p^2)$ , that is,  $\hat{D}'(p^2) = \mathcal{D}(p^2)$  where we indicate with ' the first derivative with respect to the variable  $p^2$ . The limit  $p^2 \rightarrow 0$  then yields again Eq. (17).

Clearly, one finds an IR singularity at  $p^2 = 0$ , unless  $\mathcal{D}(0) = 0$ . If this condition is satisfied, using again the trapezoidal rule, we have from Eq. (17) that

$$\lim_{p^2 \rightarrow 0} \frac{\partial \sigma(p^2)}{\partial p^2} = -\lim_{p^2 \rightarrow 0} \frac{g^2 N_c}{8\pi p^4} \frac{p^2}{2} \mathcal{D}(p^2) = -\lim_{p^2 \rightarrow 0} \frac{g^2 N_c}{16\pi} \frac{\mathcal{D}(p^2) - \mathcal{D}(0)}{p^2} = -\frac{g^2 N_c}{16\pi} \lim_{p^2 \rightarrow 0} \mathcal{D}'(p^2). \quad (19)$$

For a gluon propagator  $\mathcal{D}(p^2)$  that is regular at small momenta, i.e. that can be expanded as  $\mathcal{D}(p^2) \approx \mathcal{D}'(0)p^2 + \mathcal{D}''(0)p^4/2$  at small  $p^2$ , the above limit is finite. On the other hand, if the leading IR behavior of  $\mathcal{D}(p^2)$  is proportional to  $p^{2\eta}$  with  $1 > \eta > 0$ , as found for example in the  $2d$  case in Refs. [14, 16, 56, 59], then the above limit gives a singular value, due to the non-integer-power (and non-analytic) behavior of  $\mathcal{D}(p^2)$ .

## B. Infrared Singularity of $\sigma(p^2)$ for $d = 2$

Here we prove that — for  $d = 2$  and for any gluon propagator  $\mathcal{D}(p^2)$  that goes to zero sufficiently fast at large momenta, e.g. as  $1/p^2$ , and that is reasonably regular at small momenta, e.g. that can be expanded at  $p = 0$  as  $\mathcal{D}(p^2) \approx \mathcal{D}(0) + B p^{2\eta} + C p^{2\xi}$  (with  $\xi > \eta > 0$  and  $\mathcal{D}(0), B$  and  $C$  finite)<sup>14</sup> — the ghost form-factor (7) displays a logarithmic divergence for  $p \rightarrow 0$  proportional to  $\mathcal{D}(0)$ . Indeed, by considering Eq. (12), one obtains

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{4\pi} \left[ \int_0^p \frac{dq}{p^2} q \mathcal{D}(q^2) + \int_p^\infty \frac{dq}{q} \mathcal{D}(q^2) \right] \quad (20)$$

$$= \frac{1}{8\pi} \lim_{\Lambda \rightarrow \infty} \left\{ \int_0^{p^2} \frac{dx}{p^2} \mathcal{D}(x) + 2 \int_p^\Lambda \frac{dq}{q} \mathcal{D}(0) + 2 \int_p^\Lambda \frac{dq}{q} [\mathcal{D}(q^2) - \mathcal{D}(0)] \right\} \quad (21)$$

$$= \frac{1}{8\pi} \lim_{\Lambda \rightarrow \infty} \left\{ \int_0^{p^2} \frac{dx}{p^2} \mathcal{D}(x) + \mathcal{D}(0) \ln \left( \frac{\Lambda^2}{p^2} \right) + 2 \int_p^\Lambda \frac{dq}{q} \frac{-q^{2\eta}}{q^{2\eta} + M} \left[ \frac{\mathcal{D}(q^2) - \mathcal{D}(0)}{-q^{2\eta}} (q^{2\eta} + M) \right] \right\} \quad (22)$$

$$= \frac{1}{8\pi} \lim_{\Lambda \rightarrow \infty} \left\{ \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2} + \mathcal{D}(0) \ln \left( \frac{\Lambda^2}{p^2} \right) - \int_{p^2}^{\Lambda^2} dx \frac{x^{\eta-1}}{x^\eta + M} \left[ \frac{\mathcal{D}(x) - \mathcal{D}(0)}{-x^\eta} (x^\eta + M) \right] \right\}, \quad (23)$$

<sup>13</sup> The trapezoidal rule gives the numerical approximation

$$\int_a^b dx f(x) = \frac{b-a}{2} [f(b) + f(a)] + \mathcal{O}(b-a)^3, \quad (16)$$

which can be obtained by integrating  $f(x) \approx f(a) + (x-a)[f(b)-f(a)]/(b-a)$ . Thus, the trapezoidal rule is equivalent to using a linear Taylor expansion  $f(a) + (x-a)f'(a)$  for  $f(x)$  with the first derivative  $f'(a)$  approximated by a (first forward) finite difference  $[f(b)-f(a)]/(b-a)$ .

<sup>14</sup> For example, in Eq. (116) below, the Taylor expansion of  $\mathcal{D}(p^2)$  at  $p^2 = 0$  is of the type considered here with  $\xi = 1$ .

where  $x = q^2$ ,  $\hat{D}(x)$  is again a primitive of  $\mathcal{D}(x)$  and  $M > 0$  is a (finite) constant. If we indicate with  $H(x)$  the quantity in square brackets in the last line, then we have

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{8\pi} \lim_{\Lambda \rightarrow \infty} & \left\{ \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2} + \mathcal{D}(0) \ln \left( \frac{\Lambda^2}{p^2} \right) - \frac{1}{\eta} H(x) \ln(x^\eta + M) \right\}_{p^2}^{\Lambda^2} \\ & + \frac{1}{\eta} \int_{p^2}^{\Lambda^2} dx \ln(x^\eta + M) H'(x) \Bigg\}. \end{aligned} \quad (24)$$

Note that for  $\mathcal{D}(x) = 1/(x^\eta + M)$  we find  $H(x) = 1/M = \mathcal{D}(0)$  and the last term in Eq. (24) is zero. Since  $\lim_{x \rightarrow \infty} \mathcal{D}(x) = 0$  we also have that  $\lim_{x \rightarrow \infty} H(x) = \mathcal{D}(0)$  and the two logarithmic singularities for infinite  $\Lambda$  cancel each other. Thus, we get

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{8\pi} & \left\{ \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2} - \mathcal{D}(0) \ln(p^2) + \frac{1}{\eta} H(p^2) \ln(p^{2\eta} + M) \right. \\ & \left. + \frac{1}{\eta} \int_{p^2}^{\infty} dx \ln(x^\eta + M) \frac{\eta M [\mathcal{D}(x) - \mathcal{D}(0)] - x (x^\eta + M) \mathcal{D}'(x)}{x^{1+\eta}} \right\}. \end{aligned} \quad (25)$$

If  $\mathcal{D}(x) \sim 1/x$  at large  $x$ , it is easy to check<sup>15</sup> that  $\sigma(p^2)$  is null for  $p^2 \rightarrow \infty$ , as expected. At the same time, in the limit  $p^2 \rightarrow 0$  we obtain

$$\begin{aligned} \frac{\sigma(0)}{g^2 N_c} = \frac{1}{8\pi} & \left\{ \mathcal{D}(0) - \lim_{p^2 \rightarrow 0} \mathcal{D}(0) \ln(p^2) + \frac{1}{\eta} H(0) \ln(M) \right. \\ & \left. + \int_0^{\infty} dx \ln(x^\eta + M) \frac{\eta M [\mathcal{D}(x) - \mathcal{D}(0)] - x (x^\eta + M) \mathcal{D}'(x)}{x^{1+\eta}} \right\}, \end{aligned} \quad (26)$$

where we used

$$\lim_{p^2 \rightarrow 0} \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2} = \hat{D}'(0) = \mathcal{D}(0) \quad (27)$$

and  $H(0) = -MB$  is a finite constant.<sup>16</sup> Finally, in Appendix C we show that, under the assumptions made for the gluon propagator,<sup>17</sup> the last term on the r.h.s. of Eq. (26) is finite. Thus, the only IR singularity in the ghost form-factor  $\sigma(p^2)$  is proportional to  $-\mathcal{D}(0) \ln(p^2)$ . This result is in qualitative agreement with [89]. An IR singularity plaguing the  $2d$  calculation has also been recently obtained in Ref. [92].

An alternative (equivalent) proof<sup>18</sup> can be done by performing an integration by parts. Then, Eq. (12) becomes

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{8\pi} \left[ \int_0^{p^2} \frac{dx}{p^2} \mathcal{D}(x) + \int_{p^2}^{\infty} \frac{dx}{x} \mathcal{D}(x) \right] \quad (28)$$

$$= \frac{1}{8\pi} \left[ \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2} + \ln(x) \mathcal{D}(x) \Big|_{p^2}^{\infty} - \int_{p^2}^{\infty} dx \ln(x) \mathcal{D}'(x) \right] \quad (29)$$

$$= \frac{1}{8\pi} \left[ \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2} - \ln(p^2) \mathcal{D}(p^2) - \int_{p^2}^{\infty} dx \ln(x) \mathcal{D}'(x) \right], \quad (30)$$

<sup>15</sup> See details in Appendix C.

<sup>16</sup> Note that here we used the IR expansion  $\mathcal{D}(p^2) \approx \mathcal{D}(0) + B p^{2\eta} + C p^{2\xi}$  for the gluon propagator.

<sup>17</sup> In the same Appendix we will also show that the hypotheses considered above for the gluon propagator  $\mathcal{D}(x)$  can be relaxed.

<sup>18</sup> Note that both proofs are singling out the singularity  $-\mathcal{D}(0) \ln(p^2)$  by essentially doing a Taylor expansion of  $\mathcal{D}(p^2)$  at  $p^2 = 0$ .



where we used the assumption  $\mathcal{D}(x) \sim 1/x$  at large  $x$ . Note that the above result coincides with Eq. (25) when  $M = 0$ , which implies  $H(x) = \mathcal{D}(0) - \mathcal{D}(x)$ . A second integration by parts yields

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{8\pi} \left\{ \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2} - \ln(p^2) \mathcal{D}(p^2) - [x \ln(x) - x] \mathcal{D}'(x) \Big|_{p^2}^{\infty} + \int_{p^2}^{\infty} dx [x \ln(x) - x] \mathcal{D}''(x) \right\} \quad (31)$$

$$= \frac{1}{8\pi} \left\{ \frac{\hat{D}(p^2) - \hat{D}(0)}{p^2} - \ln(p^2) \mathcal{D}(p^2) + [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2) + \int_{p^2}^{\infty} dx [x \ln(x) - x] \mathcal{D}''(x) \right\}. \quad (32)$$

Here we used the hypothesis that  $\mathcal{D}'(x)$  goes to zero sufficiently fast at large momenta, e.g. as  $1/x^2$ . As before, one easily sees that  $\sigma(p^2)$  is null for  $p^2 \rightarrow \infty$  (see Appendix C). At the same time, under the assumptions made for the gluon propagator  $\mathcal{D}(p^2)$ , in the limit  $p^2 \rightarrow 0$  we obtain

$$\frac{\sigma(0)}{g^2 N_c} = \frac{1}{8\pi} \left\{ \mathcal{D}(0) - \lim_{p^2 \rightarrow 0} \ln(p^2) \mathcal{D}(0) + \int_0^{\infty} dx [x \ln(x) - x] \mathcal{D}''(x) \right\} \quad (33)$$

and we again find<sup>19</sup> an IR singularity proportional to  $-\mathcal{D}(0) \ln(p^2)$ , unless one has  $\mathcal{D}(0) = 0$ .

Thus, in the  $2d$  case and using a generic (sufficiently regular) gluon propagator, a null value for  $\mathcal{D}(0)$  is a necessary condition to obtain a finite value for  $\sigma(0)$  at one loop. As a consequence, the condition  $\mathcal{D}(0) = 0$  must be imposed if one wants to satisfy the no-pole condition (3) and keep the functional integration inside the first Gribov region  $\Omega$ . It is important to stress again that our proofs apply to any Gribov copy inside the first Gribov horizon, i.e. the result  $\mathcal{D}(0) = 0$  is not affected by the Gribov noise.

### C. Properties of $\sigma(p^2)$ in $d$ Dimensions: Approximate Calculation

We can easily extend the result

$$\frac{\partial \sigma(p^2)}{\partial p^2} < 0 \quad (34)$$

to the  $d$ -dimensional case by using for the integral in  $d^d q$  the so-called  $y$ -max approximation or angular approximation (see for example [13, 18, 94]). The same approach allows us to show that the IR singularity  $-\mathcal{D}(0) \ln(p^2)$  is present only in the two-dimensional case. Indeed, by using hyperspherical coordinates (see Appendix A) and by considering the positive  $x_1$  direction parallel to the external momentum  $p$ , we can write the  $d$ -dimensional ghost form-factor (7) as

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{p_\mu p_\nu}{p^2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(p-q)^2} \mathcal{D}(q^2) \mathcal{P}_{\mu\nu}(q) = \int_0^{\infty} dq \frac{q^{d-1}}{(2\pi)^d} \mathcal{D}(q^2) \int d\Omega_d \frac{1 - \cos^2(\phi_1)}{(p-q)^2}. \quad (35)$$

In the  $y$ -max approximation one substitutes  $1/(p-q)^2$  with  $1/p^2$ , for  $q^2 < p^2$ , and with  $1/q^2$ , for  $p^2 < q^2$ . Then, we obtain

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{(2\pi)^d} \left[ \int_0^p dq \frac{q^{d-1}}{p^2} \mathcal{D}(q^2) + \int_p^{\infty} dq \frac{q^{d-1}}{q^2} \mathcal{D}(q^2) \right] \int [1 - \cos^2(\phi_1)] d\Omega_d \quad (36)$$

$$= \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{2d} \left[ \int_0^{p^2} dq^2 \frac{q^{d-2}}{p^2} \mathcal{D}(q^2) + \int_{p^2}^{\infty} dq^2 \frac{q^{d-2}}{q^2} \mathcal{D}(q^2) \right] \quad (37)$$

$$= \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{2d} \int_0^{\infty} dq^2 q^{d-2} \mathcal{D}(q^2) \left[ \frac{\theta(p^2 - q^2)}{p^2} + \frac{\theta(q^2 - p^2)}{q^2} \right], \quad (38)$$

<sup>19</sup> In Appendix C we will prove that the integral on the r.h.s. of Eq. (33) is finite under the hypotheses made for the gluon propagator  $\mathcal{D}(p^2)$ . In the same Appendix we will also show how these hypotheses can be relaxed in this case.

where we used Eq. (A12). Note that, for  $d = 2$  and using Eq. (A5) one gets the exact result (14). By repeating the argument shown in the Section II A, the proof of the inequality (34) follows directly from Eq. (38).

At the same time, we can write Eq. (37) as

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \left[ \int_0^p dq q^{d-1} \frac{\mathcal{D}(q^2)}{p^2} + \int_p^\infty dq q^{d-3} \mathcal{D}(q^2) \right] = I_d(p^2, \infty), \quad (39)$$

where the integral  $I_d(p^2, \ell)$  is defined in Eq. (B34). In Appendix B we have also shown that, for  $d > 2$ , this integral is finite when the gluon propagator  $\mathcal{D}(p^2)$  is finite and nonzero at  $p^2 = 0$ . Thus, using the  $y$ -max approximation, we find that only in the  $2d$  case the condition  $\mathcal{D}(0) = 0$  is necessary in order to obtain a finite value for the Gribov form-factor  $\sigma(p^2)$  for all values of  $p^2$ .

Of course, in case of ultraviolet (UV) divergences we should regularize the integral defining  $\sigma(p^2)$ , as done for example in Section III C below for the  $4d$  case using the modified minimal subtraction ( $\overline{\text{MS}}$ ) scheme and dimensional regularization. One can also consider a fixed momentum  $\mu$  and subtract<sup>20</sup> the value  $\sigma(\mu^2)$  from the Gribov form-factor  $\sigma(p^2)$ . Due to the use of the  $y$ -max approximation the result of the subtraction is very simple. Indeed, instead of Eq. (39) we have the relation

$$\frac{\sigma(p^2) - \sigma(\mu^2)}{g^2 N_c} = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{2d} \left[ \int_0^{p^2} dx \frac{x^{d/2-1}}{p^2} \mathcal{D}(x) + \int_{p^2}^{\mu^2} dx x^{d/2-2} \mathcal{D}(x) - \int_0^{\mu^2} dx \frac{x^{d/2-1}}{\mu^2} \mathcal{D}(x) \right], \quad (40)$$

which is valid for  $p^2 \leq \mu^2$  as well as for  $\mu^2 < p^2$ . Then, we find again

$$\frac{\partial \sigma(p^2)}{\partial p^2} = -g^2 N_c \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{2d} \int_0^{p^2} dx \frac{x^{d/2-1}}{p^4} \mathcal{D}(x) < 0 \quad (41)$$

if  $\mathcal{D}(x)$  is positive. We can also easily check that, for  $\mathcal{D}(0) > 0$  and  $d > 2$ , the Gribov form-factor  $\sigma(p^2)$  in Eq. (40) does not display an IR singularity.

#### D. Properties of $\sigma(p^2)$ in $d$ Dimensions: Exact Calculation

One can improve the results obtained in the previous Section by considering the formulae reported in Appendices A and B, which allow us to perform the angular integration in Eq. (35) without approximations. Indeed, we have<sup>21</sup>

$$\frac{\sigma(p^2)}{g^2 N_c} = \int_0^\infty dq \frac{q^{d-1}}{(2\pi)^d} \mathcal{D}(q^2) \int d\Omega_d \frac{1 - \cos^2(\phi_1)}{p^2 + q^2 - 2pq \cos(\phi_1)} = I(p^2, 1, d, \infty), \quad (42)$$

with  $I(p^2, \nu, d, \ell)$  defined in Eq. (B31). Since  $\nu = 1$  in this case, for  $d \geq 2$  we can also make use of the inequalities (B36) and write

$$\frac{d}{2(d-1)} I_d(p^2, \infty) \leq I(p^2, 1, d, \infty) \leq I_d(p^2, \infty). \quad (43)$$

Note that  $I_d(p^2, \infty)$  is the same integral obtained on the right-hand side of Eq. (39). Thus, the  $y$ -max approximation of the previous Section provides, for  $d = 3$  and  $4$ , an upper bound for the Gribov ghost factor. On the contrary, for  $d = 2$ , the above inequalities become equalities. At the same time, as one can see in Appendix B, the integral  $I_d(p^2, \infty)$  is finite (for  $d > 2$ ) also if  $\mathcal{D}(0)$  is nonzero, i.e. we do not need to impose the condition  $\mathcal{D}(0) = 0$  in order to attain a finite value for  $\sigma(p^2)$  in the IR limit.

<sup>20</sup> This is equivalent to a momentum-subtraction (MOM) renormalization scheme defined by the condition  $\mathcal{G}(\mu^2) = 1/\mu^2$ .

<sup>21</sup> Again we make the hypothesis that a regularization is introduced in the case of UV divergences. In Appendix D we explicitly show how to extend the proof to a MOM scheme, i.e. by subtracting the value  $\sigma(\mu^2)$  from  $\sigma(p^2)$  where  $\mu$  is a fixed momentum.

By evaluating the derivative with respect to  $p^2$  of the result (B32) we also obtain

$$\begin{aligned} \frac{1}{g^2 N_c} \frac{\partial \sigma(p^2)}{\partial p^2} &= \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \int_0^\infty dq q^{d-1} \mathcal{D}(q^2) \left[ -\frac{\theta(p^2 - q^2)}{p^4} {}_2F_1(1, 1 - d/2; 1 + d/2; q^2/p^2) \right. \\ &\quad - \frac{q^2 \theta(p^2 - q^2)}{p^6} {}_2F_1'(1, 1 - d/2; 1 + d/2; q^2/p^2) \\ &\quad \left. + \frac{\theta(q^2 - p^2)}{q^4} {}_2F_1'(1, 1 - d/2; 1 + d/2; p^2/q^2) \right], \end{aligned} \quad (44)$$

where  ${}_2F_1'(a, b; c; z)$  indicates the derivative with respect to the variable  $z$  of the Gauss hypergeometric function  ${}_2F_1(a, b; c; z)$  (see Appendix B). Here we used again the properties of the theta and of the Dirac delta functions and Eq. (B6). For  $d = 2$ , the last two terms in Eq. (44) are null [see Eq. (B23)] and, using the result (B8), we find again Eq. (15). In the  $4d$  case one can use the expression (B9) for the Gauss hypergeometric function  ${}_2F_1(1, 1 - d/2; 1 + d/2; z)$ . Then, from Eq. (44) — or, equivalently, by evaluating the derivative with respect to  $p^2$  of Eq. (D1) in Appendix D — we find that

$$\frac{\partial \sigma(p^2)}{\partial p^2} = \frac{g^2 N_c}{32 \pi^2} \left[ \int_0^p dq \mathcal{D}(q^2) \frac{2q^5 - 3p^2 q^3}{p^6} - \int_p^\infty dq \frac{\mathcal{D}(q^2)}{q} \right] \quad (45)$$

$$= \frac{g^2 N_c}{32 \pi^2} \left[ \int_0^1 dy \mathcal{D}(y^2 p^2) (2y^5 - 3y^3) - \int_p^\infty dq \frac{\mathcal{D}(q^2)}{q} \right], \quad (46)$$

where  $y = q/p$  and we have used Eq. (A5). For  $\mathcal{D}(p^2) > 0$  both terms in square brackets are negative, i.e. the derivative  $\partial \sigma(p^2)/\partial p^2$  is negative for all values of the momentum  $p$ . Let us note that in the original work by Gribov [35] the same result was proven [see comment after Eq. (37) in the same reference] under the much stronger hypothesis of a gluon propagator  $\mathcal{D}(q^2)$  decreasing monotonically with  $q^2$  over the main range of integration.

A similar analysis can be done in the  $3d$  case using Eq. (B17). In order to simplify the notation we define

$$\Psi(z) = {}_2F_1(1, -1/2; 5/2; z) = \frac{3}{4} + \frac{3(1-z)}{8z} \left[ 1 - \frac{1-z}{\sqrt{z}} \operatorname{arcsinh} \left( \sqrt{\frac{z}{1-z}} \right) \right]. \quad (47)$$

This gives

$$\Psi'(z) = \frac{3}{16z^3} \left[ z(z-3) + \sqrt{z}(3-2z-z^2) \operatorname{arcsinh} \left( \sqrt{\frac{z}{1-z}} \right) \right]. \quad (48)$$

Then, after setting  $d = 3$  in Eq. (44) and using Eq. (A5), we obtain

$$\begin{aligned} \frac{\partial \sigma(p^2)}{\partial p^2} &= \frac{g^2 N_c}{3\pi^2} \left\{ \int_0^\infty dq q^2 \mathcal{D}(q^2) \left[ -\frac{\theta(p^2 - q^2)}{p^4} \Psi\left(\frac{q^2}{p^2}\right) - \frac{q^2 \theta(p^2 - q^2)}{p^6} \Psi'\left(\frac{q^2}{p^2}\right) \right. \right. \\ &\quad \left. \left. + \frac{\theta(q^2 - p^2)}{q^4} \Psi'\left(\frac{p^2}{q^2}\right) \right] \right\} \end{aligned} \quad (49)$$

$$= \frac{g^2 N_c}{6\pi^2} \left\{ \int_0^{p^2} dx \sqrt{x} \mathcal{D}(x) \left[ -\frac{1}{p^4} \Psi\left(\frac{x}{p^2}\right) - \frac{x}{p^6} \Psi'\left(\frac{x}{p^2}\right) \right] + \int_{p^2}^\infty \frac{dx}{x^{3/2}} \mathcal{D}(x) \Psi'\left(\frac{p^2}{x}\right) \right\}, \quad (50)$$

where we also made the substitution  $x = q^2$ . Next, the change of variable  $x = yp^2$  in the first integral and  $x = p^2/y$  in the second integral yield

$$\frac{\partial \sigma(p^2)}{\partial p^2} = \frac{g^2 N_c}{6\pi^2} \left\{ -\int_0^1 dy \frac{\sqrt{y}}{p} \mathcal{D}(yp^2) \left[ \Psi(y) + y \Psi'(y) \right] + \int_0^1 \frac{dy}{p} \frac{1}{\sqrt{y}} \mathcal{D}(p^2/y) \Psi'(y) \right\}. \quad (51)$$

As one can see in Figure 2, the factor  $-\left[\Psi(y) + y \Psi'(y)\right]$  is negative for  $y \in [0, 1]$ . At the same time, from Eq. (B24) we know that  $\Psi'(y)$  is negative for  $y \geq 0$  (see also the corresponding plot in Figure 2). Thus, for a positive gluon propagator  $\mathcal{D}(p^2)$ , the  $3d$  derivative  $\partial \sigma(p^2)/\partial p^2$  is negative for  $p^2 > 0$ .

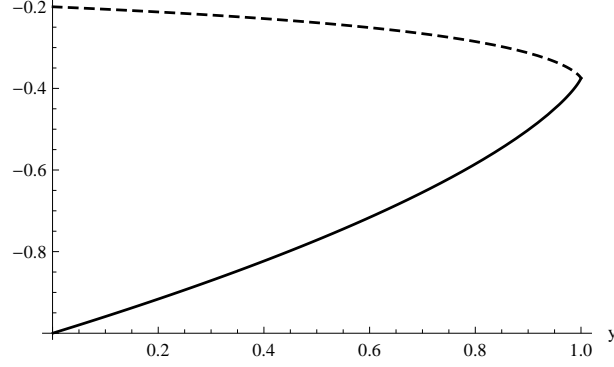


FIG. 2. The functions  $-\left[\Psi(y) + y\Psi'(y)\right]$  (full line) and  $\Psi'(y)$  (dashed line) for  $y \in [0, 1]$ .

Finally, we can consider a general  $d > 2$  and, after suitable changes of variables (for  $p^2 > 0$ ), we write

$$\frac{1}{g^2 N_c} \frac{\partial \sigma(p^2)}{\partial p^2} = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \frac{p^{d-4}}{2} \left\{ \int_0^1 dy y^{2-d/2} \mathcal{D}(p^2/y) {}_2F_1'(1, 1-d/2; 1+d/2; y) \right. \\ \left. - \int_0^1 dy y^{d/2-1} \mathcal{D}(yp^2) \left[ {}_2F_1(1, 1-d/2; 1+d/2; y) + y {}_2F_1'(1, 1-d/2; 1+d/2; y) \right] \right\}. \quad (52)$$

Note that the dependence on  $p^2$  is only in the global factor  $p^{d-4}$  and in the argument of the gluon propagator. From Appendix B we know that the derivative  ${}_2F_1'(1, 1-d/2; 1+d/2; x)$  is negative for  $x \in [0, 1]$  and  $d > 2$  and that, under the same hypotheses, the expression in square brackets is positive. Thus, for a positive gluon propagator, both term in the r.h.s. of the above expression are negative and we have proven that, for any dimension  $d \geq 2$ , the Gribov form-factor  $\sigma(p^2)$  (at one loop) is monotonically decreasing with  $p^2$ , i.e. it gets its maximum value at  $p^2 = 0$ .

### III. EVALUATION OF THE ONE-LOOP CORRECTED GHOST PROPAGATOR USING (LINEAR COMBINATIONS OF) YUKAWA-LIKE GLUON PROPAGATORS

In the previous Section we have proven that, at one-loop level and for a sufficiently regular gluon propagator  $\mathcal{D}(p^2)$ , the Gribov ghost form-factor  $\sigma(p^2)$  is always finite in three and four space-time dimensions while, in  $d = 2$ , one needs to impose  $\mathcal{D}(0) = 0$  in order to avoid an IR singularity of the type  $-\mathcal{D}(0) \ln(p^2)$ . In this Section we present an explicit calculation of  $\sigma(p^2)$  at one loop for  $d = 2, 3$  and 4 using, for the gluon propagator, results recently presented in Ref. [59, 60] from fits to lattice data of  $\mathcal{D}(p^2)$  in the SU(2) case. The expressions obtained below for the ghost propagator  $\mathcal{G}(p^2)$  will be used in a subsequent work [95] to model lattice data of SU(2) ghost propagators.

In this Section, besides recovering the same results reported in Section II, we also find that the ghost propagator  $\mathcal{G}(p^2)$  admits a one-parameter family of behaviors [21, 34] labelled by the coupling constant  $g^2$ , considered as a free parameter. The no-pole condition  $\sigma(0) \leq 1$  implies  $g^2 \leq g_c^2$ , where  $g_c^2$  is a “critical” value. Moreover, for  $g^2$  smaller than  $g_c^2$  one has  $\sigma(0) < 1$  and the ghost propagator shows a free-like behavior in the IR limit, in agreement with the so-called massive solution of gluon and ghost DSEs [18–25]. On the contrary, for  $g^2 = g_c^2$  one finds  $\sigma(0) = 1$  and the ghost propagator is IR enhanced [1, 13–16].

### A. Yukawa-Like Gluon Propagators and Set Up

In Ref. [59, 60] the SU(2) gluon propagator was fitted in 2, 3 and 4 space-time dimensions using, respectively, the functions<sup>22</sup>

$$\mathcal{D}(p^2) = C \frac{p^2 + l p^\eta + s}{p^4 + u^2 p^2 + t^2}, \quad (53)$$

$$\mathcal{D}(p^2) = C \frac{p^4 + (s+1)p^2 + s}{p^6 + (k+u^2)p^4 + (ku^2 + t^2)p^2 + kt^2} \quad (54)$$

and

$$\mathcal{D}(p^2) = C \frac{p^2 + s}{p^4 + u^2 p^2 + t^2}. \quad (55)$$

The last two propagators are tree-level gluon propagators that arise in the study of the RGZ action [82–85]. The first one is a simple generalization of the form (55) that fits well the  $2d$  data. Note that these three functions can be written as a linear combination of propagators of the type  $1/(p^2 + \omega^2)$ , where  $\omega^2$  is in general a complex number. [In the  $2d$  case we need to consider the more general form  $p^\eta/(p^2 + \omega^2)$  with  $\eta \geq 0$ .] Thus, in order to evaluate  $\sigma(p^2)$  in Eq. (7) using the above gluon propagators  $\mathcal{D}(p^2)$ , we first consider the integral

$$f(p, \omega^2) = \frac{p_\mu p_\nu}{p^2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(p-q)^2} \frac{1}{q^2 + \omega^2} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \quad (56)$$

The evaluation of  $f(p, \omega^2)$  can be done in three and four space-time dimensions by introducing Feynman parameters and applying the usual shift in the momentum  $q$ . The integration then yields

$$\begin{aligned} f(p^2, \omega^2) &= \frac{1}{(4\pi)^{d/2}} \int_0^1 dx \left[ \Delta^{d/2-2} \Gamma(2-d/2) \right] \\ &\quad - \frac{1}{(4\pi)^{d/2}} \int_0^1 dx \int_0^{1-x} dy \left[ \frac{1}{2} \Theta^{d/2-2} \Gamma(2-d/2) + x^2 p^2 \Theta^{d/2-3} \Gamma(3-d/2) \right] \end{aligned} \quad (57)$$

with

$$\Delta = -x^2 p^2 + x p^2 + (1-x)\omega^2, \quad \Theta = -x^2 p^2 + x p^2 + y \omega^2. \quad (58)$$

Since the Gamma function has the behavior  $\Gamma(x) \approx 1/x$  for small  $x$ , it is clear that the first two integrals are UV finite for  $d < 4$  while the third one is UV finite for  $d < 6$ . Below we will calculate the integral (57) for  $d = 3$  and 4. We start from the case  $d = 3$ , where all terms are finite, and then we evaluate the integral for the case  $d = 4$ , using the  $\overline{\text{MS}}$  scheme. On the contrary, as stressed above, in the  $2d$  case one needs to evaluate the more general function

$$f(p, \omega^2, \eta) = \frac{p_\mu p_\nu}{p^2} \int \frac{d^2 q}{(2\pi)^d} \frac{1}{(p-q)^2} \frac{q^\eta}{q^2 + \omega^2} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \quad (59)$$

with  $\eta \geq 0$ . This case will be treated (in a slightly different way) in Section III D.

Most of the analytic results reported in this Section have been checked using **Mathematica** and/or **Maple**.

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<sup>22</sup> Note that, for consistency with the notation used in Ref. [59], in this Section the non-integer power of the momentum  $p$  is  $\eta$  and not  $2\eta$  as in the rest of the manuscript.

### B. Ghost Propagator in the 3d Case

In the 3d case the residual  $x$ - and  $y$ -integrations in Eq. (57) are straightforward and give

$$f(p^2, \omega^2) = \left[ \frac{1}{4\pi p} \arctan\left(\frac{p}{\sqrt{\omega^2}}\right) \right] + \left[ -\frac{(p^2 - \omega^2)\sqrt{\omega^2}}{32\pi p^2 \omega^2} + \frac{\pi p}{64\pi \omega^2} - \frac{(p^2 + \omega^2)^2}{32\pi p^3 \omega^2} \arctan\left(\frac{p}{\sqrt{\omega^2}}\right) \right] \\ + \left[ \frac{3(p^2 - \omega^2)\sqrt{\omega^2}}{32\pi p^2 \omega^2} + \frac{3p^4 - 2p^2 \omega^2 + 3(\omega^2)^2}{32\pi p^3 \omega^2} \arctan\left(\frac{p}{\sqrt{\omega^2}}\right) - \frac{3\pi p}{64\pi \omega^2} \right], \quad (60)$$

where the three square brackets highlight the contribution from the three terms in Eq. (57). Here we have only made the assumption  $p^2 > 0$ . By simplifying the above result, we find

$$f(p^2, \omega^2) = \frac{1}{4\pi p} \arctan\left(\frac{p}{\sqrt{\omega^2}}\right) + \frac{(p^2 - \omega^2)\sqrt{\omega^2}}{16\pi p^2 \omega^2} - \frac{\pi p}{32\pi \omega^2} + \frac{(p^2 - \omega^2)^2}{16\pi p^3 \omega^2} \arctan\left(\frac{p}{\sqrt{\omega^2}}\right) \\ = \frac{1}{32\pi p^3 \omega^2} g(p^2, \omega^2) \quad (61)$$

where

$$g(p^2, \omega^2) = -\pi p^4 + 2p^3 \sqrt{\omega^2} - 2p(\omega^2)^{3/2} + 2(p^2 + \omega^2)^2 \arctan\left(\frac{p}{\sqrt{\omega^2}}\right). \quad (62)$$

In order to use the result (61) we need to write the gluon propagator (54) as

$$\mathcal{D}(p^2) = \frac{\alpha}{p^2 + \omega_1^2} + \frac{\beta}{p^2 + \omega_2^2} + \frac{\gamma}{p^2 + \omega_3^2}. \quad (63)$$

Here  $\omega_1^2$ ,  $\omega_2^2$  and  $\omega_3^2$  are the three roots of the cubic equation, with respect to the variable  $p^2$ , obtained by setting equal to zero the denominator of Eq. (54). Thus, by combining Eqs. (7), (63) and (56) we can write for the function  $\sigma(p^2)$  in the 3d case the expression

$$\sigma(p^2) = g^2 N_c [\alpha f(p^2, \omega_1^2) + \beta f(p^2, \omega_2^2) + \gamma f(p^2, \omega_3^2)] \quad (64)$$

or

$$\sigma(p^2) = \frac{g^2 N_c}{32\pi \omega^2 p^3} [\alpha g(p^2, \omega_1^2) + \beta g(p^2, \omega_2^2) + \gamma g(p^2, \omega_3^2)] \quad (65)$$

with  $g(p^2, \omega^2)$  given in Eq. (62). In general, the roots  $\omega_1^2$ ,  $\omega_2^2$  and  $\omega_3^2$  are all real or there is one real root, for example  $\omega_1^2$ , and two complex-conjugate roots, i.e.  $(\omega_2^2)^* = \omega_3^2$ , implying also  $\beta = \gamma^*$ . Since the fits in Ref. [59, 60] support the latter case we write

$$\beta = a + ib, \quad \gamma = a - ib \quad (66)$$

and

$$\omega_2^2 = v + iw, \quad \omega_3^2 = v - iw. \quad (67)$$

Then, following for example [96], we find for  $\omega_2^2$  the relations

$$\sqrt{\omega_2^2} = \sqrt{v + iw} = \frac{1}{\sqrt{2}} \sqrt{\sqrt{v^2 + w^2} + v} + \frac{i}{\sqrt{2}} \sqrt{\sqrt{v^2 + w^2} - v}, \quad (68)$$

$$(\omega_2^2)^{3/2} = (v + iw)^{3/2} = (v + iw) \sqrt{v + iw}, \quad (69)$$

$$\frac{p}{\sqrt{\omega_2^2}} = \frac{p}{\sqrt{v + iw}} = \frac{p}{\sqrt{v^2 + w^2}} \sqrt{v - iw} \quad (70)$$

and similar results for  $\omega_3^2$ . We also use the expression (see for example [97])

$$\arctan(z) = \frac{1}{2} \arg \left( \frac{i-z}{i+z} \right) - \frac{i}{2} \ln \left| \frac{i-z}{i+z} \right| \quad \forall z \neq \{i, -i\}. \quad (71)$$

This allows us to write the function  $\sigma(p^2)$  only in term of real quantities, i.e.

$$\sigma(p^2) = \frac{g^2 N_c}{8} \left[ \frac{\alpha g(p^2, \omega_1^2)}{4 \pi \omega_1^2 p^3} + f_R(p^2) \right] \quad (72)$$

where  $g(p^2, \omega^2)$  is given in Eq. (62) above. Also, we have

$$f_R(p^2) = f_1(p^2) + f_2(p^2) + f_3(p^2) + f_4(p^2) + f_5(p^2) \quad (73)$$

with

$$f_1(p^2) = -p \frac{av + bw}{2 R^2}, \quad (74)$$

$$f_2(p^2) = \frac{(av + bw) \sqrt{R+v} - (bv - aw) \sqrt{R-v}}{\sqrt{2} \pi R^2}, \quad (75)$$

$$f_3(p^2) = -\frac{1}{p^2} \frac{a \sqrt{R+v} - b \sqrt{R-v}}{\sqrt{2} \pi}, \quad (76)$$

$$f_4(p^2) = A(p^2) \frac{p^4 (av + bw) + 2 a p^2 R^2 + R^2 (av - bw)}{2 \pi R^2 p^3}, \quad (77)$$

$$f_5(p^2) = -L(p^2) \frac{p^4 (bv - aw) + 2 b p^2 R^2 + R^2 (bv + aw)}{2 \pi R^2 p^3} \quad (78)$$

and

$$A(p^2) = \begin{cases} \arctan \left( \frac{\sqrt{2} p \sqrt{R+v}}{R - p^2} \right) & \text{if } R - p^2 > 0 \\ \pi + \arctan \left( \frac{\sqrt{2} p \sqrt{R+v}}{R - p^2} \right) & \text{if } R - p^2 < 0 \end{cases}, \quad (79)$$

$$L(p^2) = \ln \left[ \frac{\sqrt{p^4 + 2 p^2 v + R^2}}{R + p (p + \sqrt{2} \sqrt{R-v})} \right], \quad (80)$$

$$R = \sqrt{v^2 + w^2}. \quad (81)$$

One can check that  $\sigma(p^2)$  is null in the limit  $p \rightarrow \infty$ . Finally, by expanding  $\sigma(p^2)$  around  $p^2 = 0$  in Eqs. (72)–(81) we find

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &= \frac{\alpha}{6 \pi \sqrt{\omega_1^2}} + \sqrt{R+v} \frac{9 a R^2 - (a v - b w) (2 v - R)}{24 \sqrt{2} \pi R^3} \\ &+ \sqrt{R-v} \frac{9 b R^2 - (b v + a w) (2 v + R)}{24 \sqrt{2} \pi R^3} - \frac{\alpha R^2 + 2 (a v + b w) \omega_1^2}{32 \omega_1^2 R^2} p + O(p^2), \end{aligned} \quad (82)$$

which implies  $\mathcal{G}(p^2) \propto p^{-2}$  at very small momenta. However, if the constant term in the above expression is equal to  $1/(g^2 N_c)$ , yielding  $\sigma(0) = 1$ , then one gets in the IR limit  $\mathcal{G}(p^2) \propto p^{-4}$  or  $\mathcal{G}(p^2) \propto p^{-3}$ , depending on whether the term  $\alpha R^2 + 2 (a v + b w) \omega_1^2$  vanishes or not. In particular, in the original GZ case, i.e. when the terms containing  $\omega_1^2$  and  $\alpha$  are absent, we do recover the usual  $1/p^4$  behavior. Also note that for purely imaginary poles, i.e. when  $v = b = 0$  (and  $R = w$ ), the condition  $\sigma(0) = 1$  simplifies to

$$\frac{\alpha}{\sqrt{\omega_1^2}} + \frac{\sqrt{2} a}{\sqrt{w}} = \frac{6 \pi}{g^2 N_c}. \quad (83)$$

Clearly, for a given value of  $N_c$  and with a suitable choice of  $g^2$ , one can always set  $\sigma(0) = 1$  in Eq. (82). For example, using the numerical data in the second column of Table XI of Ref. [59] and  $N_c = 2$ , we find from Eq. (82) the result<sup>23</sup>

$$\frac{\sigma(p^2)}{2g^2} \approx 0.039(0.001) - 0.017(0.003)p + O(p^2) \quad (84)$$

and we have  $\sigma(0) = 1$  if  $g^2 \approx 12.82$ . Thus, if one considers  $g^2$  as a free parameter, then Eq. (82) gives a one-parameter family of behaviors, labeled by  $g^2$ . For a specific value of  $g^2 = g_c^2$  we have  $\sigma(0) = 1$  and one finds an IR-enhanced ghost propagator at one loop. On the contrary, for  $g^2 < g_c^2$  we obtain  $\sigma(0) < 1$  and  $\mathcal{G}(p^2) \propto p^{-2}$  in the IR limit. Finally, for  $g^2 > g_c^2$  the no-pole condition  $\sigma(0) \leq 1$  is not satisfied, i.e. the ghost propagator is negative in the IR limit. These findings are in qualitative agreement with the DSE results obtained in Refs. [21, 34]. Finally, note that at small momenta the function  $\sigma(p^2)$  in the above formula (84) is decreasing as  $p^2$  increases, as expected from Section IID.

### C. Ghost Propagator in the 4d Case

We want now to evaluate  $f(p^2, \omega^2)$  in Eq. (57) for  $d = 4$ . As stressed above, in this case we have to deal with UV divergences. We do the calculation in the  $\overline{\text{MS}}$  renormalization scheme using dimensional regularization with  $d = 4 - \varepsilon$ . For the first term in Eq. (57) we have

$$\frac{(4\pi)^{2-d/2}}{16\pi^2} \int_0^1 dx \left[ \Delta^{d/2-2} \Gamma(2-d/2) \right] = \frac{1}{16\pi^2} \int_0^1 dx \left[ \frac{2}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln(\Delta) \right], \quad (85)$$

where  $\gamma_E$  is the Euler constant. Then, using the usual  $\overline{\text{MS}}$  prescription, we find

$$\frac{-1}{16\pi^2} \int_0^1 dx \ln \left[ \frac{-x^2 p^2 + x p^2 + (1-x)\omega^2}{\bar{\mu}^2} \right] = \frac{-1}{16\pi^2} \int_0^1 dx \left[ \ln \left( \frac{p^2}{\bar{\mu}^2} \right) + \ln(1-x) + \ln \left( x + \frac{\omega^2}{p^2} \right) \right] \quad (86)$$

and the  $dx$  integration yields

$$\begin{aligned} & \frac{-1}{16\pi^2} \left[ \ln \left( \frac{p^2}{\bar{\mu}^2} \right) - 2 - \frac{\omega^2}{p^2} \ln \left( \frac{\omega^2}{p^2} \right) + \left( 1 + \frac{\omega^2}{p^2} \right) \ln \left( 1 + \frac{\omega^2}{p^2} \right) \right] \\ &= \frac{-1}{16p^2\pi^2} \left[ -2p^2 + p^2 \ln \left( \frac{p^2 + \omega^2}{\bar{\mu}^2} \right) + \omega^2 \ln \left( \frac{p^2 + \omega^2}{\omega^2} \right) \right], \end{aligned} \quad (87)$$

where  $\bar{\mu}$  is the renormalization scale. For the second term in Eq. (57), which is also divergent, we first perform the  $y$  integration exactly, obtaining

$$-\frac{\Gamma(2-d/2)}{(4\pi)^{d/2} \omega^2 (d-2)} \int_0^1 dx (-x^2 p^2 + x p^2)^{d/2-1} \left[ \left( 1 + \frac{\omega^2}{x p^2} \right)^{d/2-1} - 1 \right]. \quad (88)$$

The  $\varepsilon$  expansion then gives

$$\frac{1}{32\pi^2} \int_0^1 dx (1-x) \left[ \ln \left( \frac{-x^2 p^2 + x p^2}{\bar{\mu}^2} \right) + \left( 1 + \frac{x p^2}{\omega^2} \right) \ln \left( 1 + \frac{\omega^2}{x p^2} \right) - 1 \right], \quad (89)$$

where we have already applied the  $\overline{\text{MS}}$  prescription, and after integrating in  $dx$  we find

$$\frac{1}{192p^4\omega^2\pi^2} \left\{ p^4 (p^2 + 3\omega^2) \ln \left( \frac{\omega^2}{p^2} \right) + (p^2 + \omega^2)^3 \ln \left( \frac{p^2 + \omega^2}{\omega^2} \right) + p^2 \omega^2 \left[ -7p^2 - \omega^2 + 3p^2 \ln \left( \frac{p^2}{\bar{\mu}^2} \right) \right] \right\}. \quad (90)$$

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<sup>23</sup> The errors in brackets have been evaluated using a Monte Carlo analysis with 10000 samples (see Ref. [59] for details).



Finally, the third term, which is finite, yields

$$-\frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy [x^2 p^2 \Theta^{-1}] = -\frac{1}{16\pi^2 \omega^2} \int_0^1 dx x^2 p^2 \left[ \ln \left( x + \frac{\omega^2}{p^2} \right) - \ln(x) \right] \quad (91)$$

$$= -\frac{1}{96p^4 \omega^2 \pi^2} \left[ p^2 \omega^2 (p^2 - 2\omega^2) + 2p^6 \ln \left( \frac{\omega^2}{p^2} \right) + 2(p^6 + \omega^6) \ln \left( \frac{p^2 + \omega^2}{\omega^2} \right) \right]. \quad (92)$$

By summing the three results above we ultimately find (in the  $\overline{\text{MS}}$  scheme)

$$f(p^2, \omega^2) = \frac{1}{64p^4 \omega^2 \pi^2} [f_1(p^2, \omega) + f_2(p^2, \omega^2) + f_3(p^2, \omega^2)] \quad (93)$$

with

$$f_1(p^2, \omega) = p^4 (\omega^2 - p^2) \ln \left( \frac{\omega^2}{p^2} \right), \quad (94)$$

$$f_2(p^2, \omega) = - (p^6 - p^4 \omega^2 + 3p^2 \omega^4 + \omega^6) \ln \left( \frac{p^2 + \omega^2}{\omega^2} \right), \quad (95)$$

$$f_3(p^2, \omega) = p^2 \omega^2 \left[ 5p^2 + \omega^2 + p^2 \ln \left( \frac{p^2}{\mu^2} \right) - 4p^2 \ln \left( \frac{p^2 + \omega^2}{\mu^2} \right) \right]. \quad (96)$$

As shown in Ref. [59, 60], in the  $4d$  case the fit of the gluon-propagator data is done using the expression (55). Thus, in order to use the above result (93)–(96), we need to write the gluon propagator as

$$\mathcal{D}(p^2) = \frac{\alpha_+}{p^2 + \omega_+^2} + \frac{\alpha_-}{p^2 + \omega_-^2}, \quad (97)$$

where  $\omega_{\pm}$  are the roots of the quadratic equation, with respect to the variable  $p^2$ , obtained by setting equal to zero the denominator of Eq. (55). Then, the ghost form-factor in the  $\overline{\text{MS}}$  scheme is given by

$$\sigma^{\overline{\text{MS}}}(p^2) = g^2 N_c [\alpha_+ f(p^2, \omega_+^2) + \alpha_- f(p^2, \omega_-^2)] \quad (98)$$

and we have

$$\mathcal{G}^{\overline{\text{MS}}}(p^2) = \frac{1}{p^2} \left[ 1 - \sigma^{\overline{\text{MS}}}(p^2) \right]^{-1}. \quad (99)$$

Note that the function  $\sigma^{\overline{\text{MS}}}(p^2)$  is real. From [59, 60] we know that  $\omega_{\pm}^2$  are complex-conjugate roots, i.e.  $\omega_-^2 = (\omega_+^2)^*$  and  $\alpha_- = \alpha_+^*$ . By writing  $\alpha_{\pm} = a \pm ib$  and  $\omega_{\pm}^2 = v \pm iw$  we find

$$\sigma^{\overline{\text{MS}}}(p^2) = \frac{g^2 N_c}{32\pi^2 R^2} [-p^2 t_1(p^2) + R^2 t_2(p^2) + p^{-2} t_3(p^2) - p^{-4} t_4(p^2)] \quad (100)$$

with

$$t_1(p^2) = (av + bw)[\ell_2(p^2) + \ell_3(p^2)] - (bv - aw)[a_1(p^2) - a_2(p^2)], \quad (101)$$

$$t_2(p^2) = a[5 + \ell_1(p^2) + \ell_2(p^2) + \ell_3(p^2) - 4\ell_4(p^2)] - b[a_1(p^2) - a_2(p^2) - 4a_3(p^2)], \quad (102)$$

$$t_3(p^2) = [1 - 3\ell_3(p^2)](av^3 - bwv^2 + vaw^2 - bw^3) - 3a_2(p^2)(bv^3 + awv^2 + vbw^2 + aw^3), \quad (103)$$

$$t_4(p^2) = \ell_3(p^2)(av^4 - 2wbv^3 - 2vbw^3 - aw^4) + a_2(p^2)(bv^4 + 2awv^3 + 2vaw^3 - bw^4) \quad (104)$$

and

$$\ell_1(p^2) = \ln \left( \frac{p^2}{\bar{\mu}^2} \right) , \quad (105)$$

$$\ell_2(p^2) = \ln \left( \frac{R}{p^2} \right) , \quad (106)$$

$$\ell_3(p^2) = \ln \left( \frac{\sqrt{R^2 p^4 + R^4 + 2vR^2 p^2}}{R^2} \right) , \quad (107)$$

$$\ell_4(p^2) = \ln \left( \frac{\sqrt{p^4 + 2vp^2 + R^2}}{\bar{\mu}^2} \right) , \quad (108)$$

$$a_1(p^2) = \arctan \left( \frac{w}{v} \right) , \quad (109)$$

$$a_2(p^2) = \arctan \left( \frac{wp^2}{R^2 + vp^2} \right) , \quad (110)$$

$$a_3(p^2) = \arctan \left( \frac{w}{v + p^2} \right) , \quad (111)$$

$$R = \sqrt{v^2 + w^2} . \quad (112)$$

Also note that, at large momenta, one gets

$$\sigma^{\overline{\text{MS}}}(p^2) \approx -\frac{3a g^2 N_c}{32 \pi^2} \ln \left( \frac{p^2}{\bar{\mu}^2} \right) . \quad (113)$$

Finally, by expanding  $\sigma^{\overline{\text{MS}}}(p^2)$  around  $p^2 = 0$  in Eqs. (100)–(112) we obtain

$$\begin{aligned} \frac{\sigma^{\overline{\text{MS}}}(p^2)}{g^2 N_c} = & -\frac{6a \ln \left( \frac{R}{\bar{\mu}^2} \right) - 6b \arctan \left( \frac{w}{v} \right) - 5a}{64 \pi^2} \\ & + \frac{\left[ -11 + 6 \ln \left( \frac{p^2}{R} \right) \right] (av + wb) + 6(bv - aw) \arctan \left( \frac{w}{v} \right)}{192 \pi^2 R^2} p^2 + O(p^4) . \end{aligned} \quad (114)$$

Thus, if  $\sigma^{\overline{\text{MS}}}(0) = 1$  we have that  $\mathcal{G}^{\overline{\text{MS}}}(p^2) \sim 1/p^4$  at small momenta (plus logarithmic corrections). Clearly, also in  $4d$ , we obtain a one-parameter family of behaviors, labelled by the value of  $g^2$ , and the IR-enhanced ghost propagator corresponds to the upper value of  $g^2$  allowed by the no-pole condition (3). With the numerical values reported in the second column of Table IV of Ref. [59] and  $N_c = 2$  we find<sup>24</sup>

$$\frac{\sigma^{\overline{\text{MS}}}(p^2)}{2g^2} = 0.0240(0.0007) + [-0.0082(0.0003) + 0.0060(0.0002) \ln(p^2)] p^2 \quad (115)$$

and the condition  $\sigma^{\overline{\text{MS}}}(0) = 1$  corresponds<sup>25</sup> to  $g_c^2 \approx 20.83$ . Note again the negative sign of the leading order corrections at small momenta (see Section IID).

#### D. Ghost Propagator in the 2d Case

As stressed in Section III A above, Ref. [59] has shown that the fit of the gluon-propagator data in the  $2d$  case can be done using the expression

$$\mathcal{D}(p^2) = \frac{\alpha_+ + icp^\eta}{p^2 + \omega_+^2} + \frac{\alpha_- - icp^\eta}{p^2 + \omega_-^2} , \quad (116)$$

<sup>24</sup> Again, the error in parentheses have been evaluated using a Monte Carlo analysis with 10000 samples.

<sup>25</sup> Clearly, different renormalization schemes will modify the constant term in Eq. (115) and the value of  $g_c^2$ .

where  $c$  is real,  $\alpha_- = \alpha_+^*$ ,  $\omega_-^2 = (\omega_+^2)^*$  and  $\omega_\pm$  are the roots of the quadratic equation, with respect to the variable  $p^2$ , obtained by setting equal to zero the denominator of Eq. (53). Thus, in order to evaluate the ghost form-factor  $\sigma(p^2)$  we need to consider the function  $f(p, \omega^2, \eta)$ , defined in Eq. (59) above. To this end, we can choose again the positive  $x$  direction parallel to the external momentum  $p$  and consider polar coordinates. Then, after evaluating the angular integral we find

$$f(p, \omega^2, \eta) = \frac{1}{4\pi} \left[ \int_0^p \frac{dq}{p^2} \frac{q^{1+\eta}}{q^2 + \omega^2} + \int_p^\infty \frac{dq}{q^{1-\eta}} \frac{1}{q^2 + \omega^2} \right], \quad (117)$$

valid both for  $\eta = 0$  and for  $\eta > 0$ .

In the case  $\eta = 0$  the momentum integration is straightforward giving

$$f(p, \omega^2) = \lim_{\Lambda \rightarrow \infty} \frac{1}{4\pi} \left[ \int_0^p \frac{dq}{p^2} \frac{q}{q^2 + \omega^2} + \int_p^\Lambda \frac{dq}{q} \frac{1}{q^2 + \omega^2} \right] \quad (118)$$

$$= \lim_{\Lambda \rightarrow \infty} \frac{1}{4\pi} \left\{ \frac{1}{2p^2} \ln \left( 1 + \frac{p^2}{\omega^2} \right) + \frac{1}{\omega^2} \left[ \ln(\Lambda) - \ln(p) - \frac{1}{2} \ln(\Lambda^2 + \omega^2) + \frac{1}{2} \ln(p^2 + \omega^2) \right] \right\} \quad (119)$$

$$= \frac{1}{8\pi} \left[ \frac{1}{p^2} \ln \left( 1 + \frac{p^2}{\omega^2} \right) + \frac{1}{\omega^2} \ln \left( 1 + \frac{\omega^2}{p^2} \right) \right]. \quad (120)$$

Note that the second term above blows up logarithmically in the IR limit  $p \rightarrow 0$ , in agreement with the result obtained in Section II B.

For  $\eta > 0$  the second integral in Eq. (117) can be written, after the change of variable  $t = \omega^2/(q^2 + \omega^2)$ , as

$$\int_p^\infty \frac{dq}{q^{1-\eta}} \frac{1}{q^2 + \omega^2} = \frac{1}{2(\omega^2)^{1-\eta/2}} B \left( \frac{\omega^2}{p^2 + \omega^2}; 1 - \frac{\eta}{2}, \frac{\eta}{2} \right), \quad (121)$$

where

$$B(x; a, b) = \int_0^x dt t^{a-1} (1-t)^{b-1} \quad (122)$$

is the incomplete Beta function, which is defined for  $a, b > 0$  [97], implying  $2 > \eta > 0$  in our case. For the first integral in Eq. (117) we cannot use directly the changes of variable  $v = 1/q$  and  $t = 1/(1 + \omega^2 v^2)$  because we get an incomplete Beta function (122) with  $b < 0$ . In this case it is convenient to introduce a Feynman parameter (using non-integer exponents) and write

$$\int_0^p \frac{dq}{p^2} \frac{q^3}{q^{2-\eta}} \frac{1}{q^2 + \omega^2} = \frac{1}{p^2} \left( 1 - \frac{\eta}{2} \right) \int_0^1 dx x^{-\eta/2} \int_0^p \frac{q^3 dq}{[q^2 + (1-x)\omega^2]^{2-\eta/2}} \quad (123)$$

$$= \frac{1}{2p^2} \int_0^1 dx x^{-\eta/2} \left\{ -\frac{p^2}{[p^2 + (1-x)\omega^2]^{1-\eta/2}} + \frac{2}{\eta} [p^2 + (1-x)\omega^2]^{\eta/2} - \frac{2}{\eta} [(1-x)\omega^2]^{\eta/2} \right\}, \quad (124)$$

where we have also done the integration in  $dq$ . After suitable changes of variables, the last formula can be written as

$$\begin{aligned} \int_0^p \frac{dq}{p^2} \frac{q^3}{q^{2-\eta}} \frac{1}{q^2 + \omega^2} &= -\frac{1}{2(\omega^2)^{1-\eta/2}} B \left( \frac{\omega^2}{p^2 + \omega^2}; 1 - \frac{\eta}{2}, \frac{\eta}{2} \right) \\ &+ \frac{p^2 + \omega^2}{\eta p^2 (\omega^2)^{1-\eta/2}} B \left( \frac{\omega^2}{p^2 + \omega^2}; 1 - \frac{\eta}{2}, 1 + \frac{\eta}{2} \right) - \frac{1}{\eta p^2 (\omega^2)^{-\eta/2}} B \left( 1 - \frac{\eta}{2}, 1 + \frac{\eta}{2} \right), \end{aligned} \quad (125)$$

where  $B(a, b) = B(1; a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$  is the Beta function. Thus, by summing the two results above, we find

$$f(p, \omega^2, \eta) = \frac{(\omega^2)^{\eta/2}}{4\pi\eta p^2} \left[ \frac{p^2 + \omega^2}{\omega^2} B\left(\frac{\omega^2}{p^2 + \omega^2}; 1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) - B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) \right] \quad (126)$$

$$= \frac{(\omega^2)^{\eta/2-1}}{4\pi\eta} B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) + \frac{(\omega^2)^{\eta/2}}{4\pi\eta p^2} \frac{p^2 + \omega^2}{\omega^2} \left[ B\left(\frac{\omega^2}{p^2 + \omega^2}; 1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) - B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) \right]. \quad (127)$$

Note that for  $p = 0$  the incomplete Beta function  $B(\omega^2/(p^2 + \omega^2); 1 - \eta/2, 1 + \eta/2)$  becomes the Beta function  $B(1 - \eta/2, 1 + \eta/2)$ . Then, by Taylor expanding  $f(p, \omega^2, \eta)$  for small momenta  $p$ , we obtain

$$f(p, \omega^2, \eta) = \frac{(\omega^2)^{\eta/2-1}}{4\pi\eta} B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) - \frac{p^\eta}{4\pi\eta(1 + \eta/2)\omega^2} [1 - O(p^2)], \quad (128)$$

yielding a constant contribution at  $p = 0$ .

Using the expression (116), the ghost form-factor in the  $2d$  case is given by

$$\sigma(p^2) = g^2 N_c [\alpha_+ f(p^2, \omega_+^2) + \alpha_- f(p^2, \omega_-^2) + icf(p^2, \omega_+^2, \eta) - icf(p^2, \omega_-^2, \eta)], \quad (129)$$

with  $f(p^2, \omega^2)$  and  $f(p^2, \omega^2, \eta)$  defined, respectively, in Eqs. (120) and (127). Of course, the function  $\sigma(p^2)$  is real. By writing  $\alpha_\pm = a \pm ib$  and  $\omega_\pm^2 = v \pm iw$  we get for the first two terms above

$$\begin{aligned} \alpha_+ f(p^2, \omega_+^2) + \alpha_- f(p^2, \omega_-^2) &= \frac{1}{8\pi} \left\{ \frac{1}{p^2} [a \ell_3(p^2) + b a_2(p^2)] \right. \\ &\quad \left. + \frac{1}{R^2} [(av + bw) \ell_5(p^2) - (bv - aw) a_3(p^2)] \right\}, \end{aligned} \quad (130)$$

where  $\ell_3(p^2)$ ,  $a_2(p^2)$ ,  $a_3(p^2)$  and  $R$  have already been defined in Eqs. (107), (110), (111) and (112) and

$$\ell_5(p^2) = \ln \left( \frac{\sqrt{p^4 + 2vp^2 + R^2}}{p^2} \right). \quad (131)$$

As shown in Section II B, there is a logarithmic singularity  $\ell_5(p^2) \sim -\ln(p^2)$  at small momenta proportional to the gluon propagator at zero momentum, that is,  $\mathcal{D}(0) = 2(av + bw)/R^2$ . We also have

$$icf(p^2, \omega_+^2, \eta) - icf(p^2, \omega_-^2, \eta) = -2c \Im [f(p^2, \omega_+^2, \eta)], \quad (132)$$

where we have indicated with  $\Im$  the imaginary part of the expression in square brackets.

One can easily check that  $\sigma(p^2)$  is null at large momenta. Finally, the results (130) and (132), together with the expressions (128) and (129), allow us to evaluate the behavior of the ghost propagator at small momenta. We obtain

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &= \frac{1}{8\pi} \left\{ \frac{ap^2}{2R^2} + \frac{av + bw}{R^2} \left[ 1 + \ln \left( \frac{R}{p^2} \right) \right] - \frac{bv - aw}{R^2} \left[ \arctan \left( \frac{w}{v} \right) - \frac{wp^2}{R^2} \right] + O(p^4) \right\} \\ &\quad - 2c \Im \left[ \frac{(\omega_+^2)^{\eta/2-1}}{4\pi\eta} B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) - \frac{p^\eta}{4\pi\eta(1 + \eta/2)\omega_+^2} + O(p^{2+\eta}) \right] \end{aligned} \quad (133)$$

$$\begin{aligned} &= \frac{1}{8\pi} \left\{ \frac{ap^2}{2R^2} + \frac{av + bw}{R^2} \left[ 1 + \ln \left( \frac{R}{p^2} \right) \right] - \frac{bv - aw}{R^2} \left[ \arctan \left( \frac{w}{v} \right) - \frac{wp^2}{R^2} \right] \right\} \\ &\quad - 2c \sin \left[ \left( \frac{\eta}{2} - 1 \right) \arctan \left( \frac{w}{v} \right) \right] \frac{R^{\eta/2-1}}{4\pi\eta} B\left(1 - \frac{\eta}{2}, 1 + \frac{\eta}{2}\right) - \frac{2cw p^\eta}{4\pi\eta(1 + \eta/2)R^2} + O(p^{2+\eta}). \end{aligned} \quad (134)$$

Note that, if  $\sigma(0) = 1$ , one finds a ghost propagator with a behavior  $1/p^{2+\eta}$  in the IR limit. As in  $3d$  and in  $4d$  we have a one parameter family of solutions labelled by the value of  $g^2$ .

As explained in Ref. [59], the  $2d$  data for the gluon propagator suggest the relations  $a = -b$  and  $v = w$ , implying  $av + bw = 0$  and  $R^2 = 2v^2$ . Then, we find

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{a}{32v} - 2c \sin \left[ \left( \frac{\eta}{2} - 1 \right) \frac{\pi}{4} \right] \frac{(2v^2)^{\eta/2-1}}{4\pi\eta} B \left( 1 - \frac{\eta}{2}, 1 + \frac{\eta}{2} \right) - \frac{cp^\eta}{4\pi\eta(1+\eta/2)v} - \frac{ap^2}{32\pi v^2} + O(p^{2+\eta}) . \quad (135)$$

Using the approximate result  $\eta \approx 1$  (see again Ref. [59]) this formula simplifies to

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{a + 4c\sqrt{1-1/\sqrt{2}}}{32v} - \frac{cp}{6\pi v} - \frac{ap^2}{32\pi v^2} + O(p^{2+\eta}) . \quad (136)$$

On the contrary, for  $N_c = 2$  and with the numerical values reported in [59] — see the second column of Table XIV and, for the exponent  $\eta$ , the last line of Table XIII — we find for Eq. (134) the numerical results

$$\frac{\sigma(p^2)}{2g^2} \approx 0.029(0.004) - 0.029(0.005)p^{0.909(0.049)} - 0.023(0.004)p^2 . \quad (137)$$

The coefficient  $(av + bw)/R^2 \propto \mathcal{D}(0)$ , multiplying the logarithmic IR singularity, is zero within error and we have omitted the corresponding term. Note that  $\sigma(p^2)$  decreases for increasing momenta  $p^2$ , as proven in Section II A above. Also note that we have  $\sigma(0) = 1$  for  $g_c^2 \approx 17.24$  and in this case the ghost propagator behaves as  $\sim 1/p^{2.9}$  in the IR limit.

#### IV. THE GHOST PROPAGATOR BEYOND PERTURBATION THEORY

The one-loop analysis above has shown that, in the  $2d$  case, an IR singularity  $-\mathcal{D}(0) \ln(p^2)$  appears in the Gribov form-factor  $\sigma(p^2)$  when  $p^2 \rightarrow 0$ . Thus, one needs a null gluon propagator at zero momentum in order to satisfy the no-pole condition  $\sigma(0) \leq 1$ . On the contrary, for  $d = 3$  and  $4$ , we found that  $\sigma(p^2)$  is finite also for  $\mathcal{D}(p^2) > 0$ .

In this section we improve our analysis by considering the DSE for the ghost propagator  $\mathcal{G}(p^2)$  (see for example [13, 18, 94]). As stressed in the Introduction, here we do not try to solve the ghost propagator DSE, but instead we concentrate on general properties of this equation for different space-time dimensions. In particular, the results obtained in Section II are confirmed by considering a generic (sufficiently regular) gluon propagator  $\mathcal{D}(p^2)$  and an IR-finite ghost-gluon vertex  $igf^{adc}p_\lambda\Gamma_{\lambda\nu}(p, q)$ .

##### A. The $2d$ Case

In the  $2d$  Landau gauge the DSE for the ghost propagator is written as

$$\frac{1}{\mathcal{G}(p^2)} = p^2 - g^2 N_c \int \frac{d^2q}{(2\pi)^2} p_\lambda \Gamma_{\lambda\nu}(p, q) s_\mu \mathcal{D}(q^2) P_{\mu\nu}(q) \mathcal{G}(s^2) , \quad (138)$$

where  $s = p - q$ , the gluon and the ghost propagators — respectively  $\mathcal{D}(p^2)$  and  $\mathcal{G}(p^2)$  — are full propagators and we indicated with  $igf^{adc}p_\lambda\Gamma_{\lambda\nu}(p, q)$  the full ghost-gluon vertex. The above result implies

$$\sigma(p^2) = \frac{g^2 N_c}{p^2} \int \frac{d^2q}{(2\pi)^2} p_\lambda \Gamma_{\lambda\nu}(p, q) s_\nu \mathcal{D}(q^2) P_{\mu\nu}(q) \frac{1}{s^2} \frac{1}{1 - \sigma(s^2)} \quad (139)$$

if one uses Eq. (8). For a tree-level ghost-gluon vertex  $\Gamma_{\lambda\nu}(p, q) = \delta_{\lambda\nu}$  and using the transversality of the gluon propagator we finally find

$$\sigma(p^2) = g^2 N_c \frac{p_\mu p_\nu}{p^2} \int \frac{d^2q}{(2\pi)^2} \mathcal{D}(q^2) P_{\mu\nu}(q) \frac{1}{s^2} \frac{1}{1 - \sigma(s^2)} , \quad (140)$$

which should be compared to the one-loop result (7). As in Section II A above, we can choose the  $x$  direction along the external momentum  $p$  obtaining (using polar coordinates)

$$\frac{\sigma(p^2)}{g^2 N_c} = \int_0^\infty \frac{q dq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2 [1 - \sigma(s^2)]}, \quad (141)$$

with  $s^2 = p^2 + q^2 - 2pq \cos(\theta)$ .

This equation will be analyzed below using two different approaches. A first result can, however, be easily obtained using again the  $y$ -max approximation, as in Section II C above. This gives us

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{8\pi} \left\{ \int_0^{p^2} dx \frac{\mathcal{D}(x)}{p^2 [1 - \sigma(p^2)]} + \int_{p^2}^\infty dx \frac{\mathcal{D}(x)}{x [1 - \sigma(x)]} \right\}, \quad (142)$$

where we have done the angular integration and set  $x = q^2$ . In the limit of small momenta  $p^2$  we then obtain

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{8\pi} \left\{ \lim_{p^2 \rightarrow 0} \frac{p^2}{2} \frac{\mathcal{D}(p^2) + \mathcal{D}(0)}{p^2 [1 - \sigma(p^2)]} + \int_0^\infty dx \frac{\mathcal{D}(x)}{x [1 - \sigma(x)]} \right\}. \quad (143)$$

In order to avoid IR singularities in the above equation we have to impose  $\mathcal{D}(p^2) \approx Bp^{2\eta}$ , i.e. the gluon propagator should be null at zero momentum. In particular, if  $\sigma(0) < 1$ , i.e. for a free-like ghost propagator at small momenta, it is sufficient to have  $\eta > 0$ . On the contrary, if the ghost propagator is IR enhanced and  $1 - \sigma(0) \propto x^\kappa$  for small  $x$  with  $\kappa > 0$ , then the condition  $\eta > \kappa$  should be satisfied. Note that the predictions of the scaling solution [14–16], i.e.  $\eta = 0.4$  and  $\kappa = 0.2$ , are consistent with the above inequality. The same results can also be obtained by setting  $p^2 = 0$  directly in Eq. (141). This makes the  $\theta$  integral trivial and gives

$$\frac{\sigma(0)}{g^2 N_c} = \int_0^\infty \frac{q dq}{4\pi} \frac{\mathcal{D}(q^2)}{q^2 [1 - \sigma(q^2)]}. \quad (144)$$

Note, however, that in both cases we essentially miss the logarithmic IR singularity  $-\ln(p^2)$  which is found below. In the first case this is probably related to the very crude  $y$ -max approximation. On the contrary, in Eq. (144), this is due to the (improper) exchange of the  $q$  integration with the  $p^2 \rightarrow 0$  limit [89].

### 1. Bounds on the Gribov Form-Factor

Since the Gribov form-factor is non-negative, we can easily construct a lower bound for the l.h.s. of Eq. (141) by writing

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &\geq \int_0^\infty \frac{q dq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2} = I(p^2, 1, 2, \infty) \\ &= I_2(p^2, \infty) = \frac{1}{4\pi} \left[ \int_0^p \frac{dq}{p^2} q \mathcal{D}(q^2) + \int_p^\infty \frac{dq}{q} \mathcal{D}(q^2) \right], \end{aligned} \quad (145)$$

where we use the definitions (B31), (B34) and the relations (B36). The last integral in the above equation has already been analyzed in Section II B, where it was shown that  $I_2(p^2, \infty)$  develops an IR singularity proportional to  $-\ln(p^2)$  if  $\mathcal{D}(0) \neq 0$ . Thus,  $\sigma(p^2)$  also is IR singular, unless  $\mathcal{D}(0) = 0$ .

One can also find an upper bound for  $\sigma(p^2)$  and check that the IR singularity is indeed only logarithmic. To this end we can notice that, if  $\sigma(0) < 1$ , one can write<sup>26</sup>

$$\frac{\sigma(p^2)}{g^2 N_c} \leq \int_0^\infty \frac{q dq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2 [1 - \sigma(0)]} = \frac{I_2(p^2, \infty)}{1 - \sigma(0)}, \quad (146)$$

<sup>26</sup> Recall that, in the  $2d$  case and in the one-loop approximation, the function  $\sigma(p^2)$  is decreasing as  $p^2$  increases, i.e. the maximum value of  $\sigma(p^2)$  is obtained for  $p^2 = 0$  (see Section II A). However, the proof presented here can be easily modified for the case when  $\sigma(p^2) < 1$  for all momenta  $p$  and the maximum value of  $\sigma(p^2)$  is not attained at  $p = 0$ . Finally, one should note that in the DSE (140) one uses explicitly Eq. (8). Thus, when estimating the integral in Eq. (141), we cannot simply impose  $\sigma(p^2) < +\infty$  but we have to consider the stronger condition  $\sigma(p^2) \leq 1$ .

where we have also used Eqs. (145) and (145) above. Therefore, the upper bound also blows up as  $-\ln(p^2)$  in the IR limit. At the same time, if  $\sigma(0) = 1$ , with  $\sigma(p^2) \approx 1 - cp^{2\kappa}$  at small momenta we find

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &= \int_0^\infty \frac{q dq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{c s^{2+2\kappa}} \\ &\quad + \int_0^\infty \frac{q dq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2} \left[ \frac{1}{1 - \sigma(s^2)} - \frac{1}{c s^{2\kappa}} \right]. \end{aligned} \quad (147)$$

Note that the quantity in square brackets in the last integral is finite at  $s = 0$  if the behavior of  $\sigma(p^2)$  is given by  $1 - cp^{2\kappa} + \mathcal{O}(p^\tau)$  with  $\tau \geq 4\kappa$ . Moreover, this quantity goes to 1 at large momenta and its absolute value is clearly bounded from above by some positive constant  $M$  if  $\sigma(p^2) \in [0, 1]$ . Hence, we have

$$\frac{\sigma(p^2)}{g^2 N_c} \leq \int_0^\infty \frac{q dq}{4\pi^2} \mathcal{D}(q^2) \int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{c s^{2+2\kappa}} + M I_2(p^2, \infty) = \frac{1}{c} I(p^2, 1 + \kappa, 2, \infty) + M I_2(p^2, \infty). \quad (148)$$

For  $1/2 > \kappa$  we can also use the upper bound in Eq. (B33) and write

$$\frac{\sigma(p^2)}{g^2 N_c} \leq \left( \frac{M''}{c} + M \right) I_2(p^2, \infty), \quad (149)$$

where  $M''$  is a positive constant. Thus, we have again an IR singularity proportional to  $-\ln(p^2)$  if  $\mathcal{D}(0)$  is not zero. We conclude that  $\sigma(p^2)$  can be finite solely if  $\mathcal{D}(0) = 0$ .

Let us remark that the only hypothesis considered in this case is the IR expansion  $\sigma(p^2) = 1 - cp^{2\kappa} + \mathcal{O}(p^\tau)$  with  $1 > 2\kappa$  and  $\tau \geq 4\kappa$ . Also note that the  $2d$  lattice data [56] show for the ghost propagator an IR behavior in good agreement with the so-called scaling solution [14–16] that predicts  $\kappa = 0.2$ . Thus, the condition  $1 > 2\kappa$  is verified in both cases. One can also note that, by considering in Eq. (139) the full ghost-gluon vertex  $\Gamma_{\lambda\nu}(p, q)$ , instead of the tree-level one  $\delta_{\lambda\nu}$ , the above results still applies for an IR-finite vertex. This hypothesis is usually adopted in DSE studies of gluon and ghost propagators [12, 20, 25, 31] and it is confirmed by lattice data [98–101].

## 2. Analysis of the Gribov Form-Factor Using a Spectral Representation

In this section we analyze the DSE (141) in an alternative way, also avoiding the  $y$ -max approximation. To this end, let us first consider the  $\theta$ -integral using contour integration. After setting  $z = e^{i\theta}$  we find

$$\int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{p^2 [1 - \sigma(p^2)]} = \frac{i}{4} \oint dz \frac{(z^2 - 1)^2}{z^2 (-q + kz)(k - qz)} \frac{1}{1 - \sigma[(-q + kz)(k - qz)z^{-1}]}, \quad (150)$$

where the integral  $\oint dz$  is again taken on the unit circle  $|z| = 1$ . Clearly, besides the poles at  $q = k/z$  and at  $q = kz$  in the first denominator on the r.h.s. of the above equation, one has to consider possible divergences in the function

$$f(z) \equiv \frac{1}{1 - \sigma[(-q + kz)(k - qz)z^{-1}]} \quad (151)$$

In particular, if we assume ghost enhancement, i.e.  $\sigma(0) = 1$ , then  $f(z)$  is divergent at  $z = q/k$  and at  $z = k/q$ . Note that these divergences are not necessarily poles of the function  $f(z)$ . Indeed,  $f(z)$  could display a branch cut in the unit disc or one passing through it. For example, the usual  $d = 2$  DSE scaling solution has  $\mathcal{G}(k^2) \sim 1/(k^2)^\nu$  in the limit  $k^2 \rightarrow 0$ , where  $\nu$  is a fractional number. This behavior signals a non-analyticity for  $\mathcal{G}(k^2)$  at the origin and implies a non-analyticity for the function  $f(z)$  at  $z = k/q$  or at  $z = q/k$ . Also, since the ghost is “massless” we should expect that the ghost propagator develops a branch cut along the real axis for  $k^2 < 0$ . Then,  $z = q/k$  or  $z = k/q$  would correspond to branch points of the function  $f(z)$ , making quite difficult the evaluation of the contour integral in the above expression.

In order to overcome this problem, we make the hypothesis that a spectral representation for the ghost propagator can be introduced, i.e. we write<sup>27</sup>

$$\mathcal{G}(p^2) = \frac{1}{p^2} \frac{1}{1 - \sigma(p^2)} = \int_0^\infty dt \frac{\rho(t)}{t + p^2}, \quad (152)$$

which reproduces the branch cut in  $\mathcal{G}(k^2)$  for  $k^2 < 0$  (see for example [102]). If we assume  $\sigma(\infty) = 0$  and write

$$\mathcal{G}(p^2) = \frac{1}{p^2} \int_0^\infty dt \frac{\rho(t)}{1 + t/p^2} \quad (153)$$

it is clear that the spectral density  $\rho(t)$  must satisfy the normalization condition

$$1 = \int_0^\infty dt \rho(t). \quad (154)$$

Also note that the tree-level ghost propagator  $\mathcal{G}(p^2) = 1/p^2$  corresponds to the spectral density  $\rho(t) = 2\delta(t)$ , where  $\delta(t)$  is the Dirac delta function. This case will be used below to recover results obtained in the one-loop analysis carried on in Sections II A and II B. In the general case, the spectral density  $\rho(t)$  is proportional to the discontinuity of the ghost propagator along the cut.<sup>28</sup>

Considering Eqs. (141) and (152) we can write

$$\int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2 [1 - \sigma(s^2)]} = \frac{i}{4} \int_0^\infty dt \rho(t) \oint dz \frac{z^2 + \bar{z}^2 - 2}{-pqz^2 + (p^2 + q^2 + t)z - pq} \quad (155)$$

$$= -\frac{i}{4pq} \int_0^\infty dt \rho(t) \oint \frac{dz}{z^2} \frac{(z^2 - 1)^2}{z^2 - (p^2 + q^2 + t)z / (pq) + 1}, \quad (156)$$

where we indicated with  $\bar{z}$  the complex-conjugate of  $z = e^{i\theta}$ . Thus, using the representation (156) we can avoid dealing directly with the integral of an unknown function along the branch cut. In exchange, we have in our formulae an extra integration of the (also unknown) spectral density  $\rho(t)$ . Nevertheless, as we will see below, the above equation will allow us to control the  $p^2 \rightarrow 0$  limit [at least in the case  $\rho(t) > 0$ ]. To this end, let us first note that in the contour integral (156) there is a double pole at  $z = 0$  and there are single poles at

$$z_{\pm} = \frac{(p^2 + q^2 + t) \pm \sqrt{(p^2 + q^2 + t)^2 - 4p^2q^2}}{2pq}. \quad (157)$$

Since  $p, q, t \geq 0$  we have that  $p^2 + q^2 + t \geq 2pq \geq 0$  and one can check that the pole  $z_-$  lies within the unit disc while  $z_+$  lies outside of it. Moreover, for  $p^2 + q^2 + t = 2pq$  (which implies  $t = 0$  and  $p = q$ ) the two poles coincide and we have  $z_{\pm} = 1$ . It is also easy to check that the residues, inside the unit circle, for the  $z$ -integrand are

$$\mathcal{R}_{\text{es}_{z=0}} = -\frac{p^2 + q^2 + t}{p^2q^2}, \quad (158)$$

$$\mathcal{R}_{\text{es}_{z=z_-}} = \frac{\sqrt{[(p+q)^2 + t][(p-q)^2 + t]}}{p^2q^2}. \quad (159)$$

Then, using the residue theorem, we find

$$\int_0^{2\pi} d\theta \frac{1 - \cos^2(\theta)}{s^2 [1 - \sigma(s^2)]} = \frac{\pi}{2} \int_0^\infty dt \rho(t) \frac{p^2 + q^2 + t - \sqrt{[(p+q)^2 + t][(p-q)^2 + t]}}{p^2q^2} \quad (160)$$

<sup>27</sup> Since we are working in the  $d = 2$  case, the theory should be UV finite and we do not need to consider renormalization factors here.

<sup>28</sup> Note that, if  $\mathcal{G}(k^2)$  has a branch cut along a curve  $\mathcal{C}$  in the complex plane and if it goes to zero sufficiently fast at infinity, using Cauchy's theorem we could write down an integral relation similar to Eq. (152) with the variable  $t$  running over the curve  $-\mathcal{C}$ , with  $z \in -\mathcal{C} \Leftrightarrow -z \in \mathcal{C}$ . Also, possible poles can be included by adding  $\delta$ -functions to the spectral density  $\rho(t)$  or, equivalently, by pulling the pole terms out of the spectral integral.



and we can write the ghost DSE (141) as

$$\frac{\sigma(p^2)}{g^2 N_c} = \int_0^\infty \frac{q dq}{8\pi} \mathcal{D}(q^2) \int_0^\infty dt \rho(t) \frac{p^2 + q^2 + t - \sqrt{[(p+q)^2 + t][(p-q)^2 + t]}}{p^2 q^2} \quad (161)$$

$$= \int_0^\infty \frac{dx}{16\pi} \mathcal{D}(x) \int_0^\infty dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2 x}. \quad (162)$$

Note that, for  $\rho(t) = 2\delta(t)$  and using

$$\sqrt{[(p+q)^2][(p-q)^2]} = \begin{cases} p^2 - q^2 & \text{if } p^2 > q^2 \\ q^2 - p^2 & \text{if } q^2 > p^2 \end{cases} \quad (163)$$

we find from Eq. (160) the one-loop result (11). Also note that, by Taylor expanding the integrand at  $p^2 = 0$ , one finds

$$\frac{\sigma(0)}{g^2 N_c} = \frac{1}{8\pi} \int_0^\infty dx \mathcal{D}(x) \int_0^\infty dt \frac{\rho(t)}{t+x} = \frac{1}{8\pi} \int_0^\infty dx \frac{\mathcal{D}(x)}{x [1 - \sigma(x)]}, \quad (164)$$

where we used the definition (152). As shown above [see Eq. (144)], this result can also be obtained immediately by setting  $p^2 = 0$  in Eq. (141). However, as already pointed out below Eq. (144) and in Ref. [89], one should not exchange the  $q$  integration and the  $p^2 \rightarrow 0$  limit. Therefore, in order to properly evaluate  $\sigma(p^2)$  for small momenta  $p^2$ , we write Eq. (162) as

$$\begin{aligned} \frac{\sigma(0)}{g^2 N_c} &= \lim_{p^2 \rightarrow 0} \int_0^{p^2} \frac{dx}{16\pi} \mathcal{D}(x) \int_0^\infty dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2 x} \\ &\quad + \lim_{p^2 \rightarrow 0} \int_{p^2}^\infty \frac{dx}{16\pi} \mathcal{D}(x) \int_0^\infty dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2 x}. \end{aligned} \quad (165)$$

The first integral can be estimated using the the trapezoidal rule. We then obtain

$$\lim_{p^2 \rightarrow 0} \int_0^{p^2} \frac{dx}{16\pi} \mathcal{D}(x) \int_0^\infty dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2 x} \quad (166)$$

$$= \lim_{p^2 \rightarrow 0} \frac{p^2}{32\pi} \int_0^\infty dt \rho(t) \left[ \mathcal{D}(p^2) \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^4} + \frac{2\mathcal{D}(0)}{p^2 + t} \right] \quad (167)$$

$$= \lim_{p^2 \rightarrow 0} \frac{1}{32\pi} \int_0^\infty dt \rho(t) \left[ \mathcal{D}(p^2) \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} \right] + \lim_{p^2 \rightarrow 0} \frac{\mathcal{D}(0)}{16\pi [1 - \sigma(p^2)]}, \quad (168)$$

where we used again Eq. (152). For the second integral we define

$$\mathcal{G}(x, p^2) = \frac{\mathcal{D}(x)}{16\pi} \int_0^\infty dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2} \quad (169)$$

and find

$$\int_{p^2}^\infty dx \frac{\mathcal{G}(x, p^2)}{x} = \ln(x) \mathcal{G}(x, p^2) \Big|_{p^2}^\infty - \int_{p^2}^\infty dx \ln(x) \mathcal{G}'(x, p^2) \quad (170)$$

$$= -\ln(p^2) \frac{\mathcal{D}(p^2)}{16\pi} \int_0^\infty dt \rho(t) \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} - \int_{p^2}^\infty dx \ln(x) \mathcal{G}'(x, p^2), \quad (171)$$

where ' refers to the derivative w.r.t. the  $x$  variable and we used the fact that  $\mathcal{D}(x)$  goes to zero at large  $x$ . Note that, in the one-loop case  $\rho(t) = 2\delta(t)$ , we have  $\mathcal{G}(x, p^2) = \mathcal{D}(x)/(8\pi)$  and Eq. (171) becomes

$$\int_{p^2}^{\infty} \frac{dx}{8\pi} \frac{\mathcal{D}(x)}{x} = -\ln(p^2) \frac{\mathcal{D}(p^2)}{8\pi} - \int_{p^2}^{\infty} \frac{dx}{8\pi} \ln(x) \mathcal{D}'(x), \quad (172)$$

in agreement with Eqs. (28) and (30).

By collecting the above results we can therefore write

$$\begin{aligned} \frac{\sigma(0)}{g^2 N_c} = \frac{1}{16\pi} \lim_{p^2 \rightarrow 0} \left\{ \frac{\mathcal{D}(0)}{1 - \sigma(p^2)} + \mathcal{D}(p^2) \left[ \frac{1}{2} - \ln(p^2) \right] \int_0^{\infty} dt \rho(t) \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} \right. \\ \left. - \int_{p^2}^{\infty} dx \ln(x) \mathcal{G}'(x, p^2) \right\}. \end{aligned} \quad (173)$$

We can now verify that the last integral in the above expression is finite. Indeed, we have

$$\begin{aligned} \mathcal{G}'(x, p^2) = \frac{\mathcal{D}'(x)}{16\pi} \int_0^{\infty} dt \rho(t) \frac{p^2 + x + t - \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}}{p^2} \\ + \frac{\mathcal{D}(x)}{16\pi} \int_0^{\infty} dt \rho(t) \left[ \frac{1}{p^2} - \frac{x + t - p^2}{p^2 \sqrt{t^2 + 2t(p^2 + x) + (p^2 - x)^2}} \right]. \end{aligned} \quad (174)$$

Then, for large  $x$  we find

$$\mathcal{G}'(x, p^2) \sim \frac{\mathcal{D}'(x)}{8\pi} \int_0^{\infty} dt \rho(t) + \frac{\mathcal{D}(x)}{8\pi x^2} \int_0^{\infty} dt \rho(t) t = \frac{1}{8\pi} \left[ \mathcal{D}'(x) + \frac{\mathcal{D}(x)}{x^2} \int_0^{\infty} dt \rho(t) t \right], \quad (175)$$

where we used the normalization condition (154), while for small  $x$  we have

$$\mathcal{G}'(x, p^2) \sim \frac{x \mathcal{D}'(x)}{8\pi} \int_0^{\infty} dt \frac{\rho(t)}{t + p^2} + \frac{\mathcal{D}(x)}{8\pi} \int_0^{\infty} dt \frac{\rho(t)}{t + p^2} = \frac{\mathcal{G}(p^2)}{8\pi} [x \mathcal{D}'(x) + \mathcal{D}(x)], \quad (176)$$

where we used the definition (153). Thus, the integral  $\int_0^{\infty} dx \ln(x) \mathcal{G}'(x, p^2)$  has no IR and UV singularities if  $\mathcal{D}(x)$  and  $\mathcal{D}'(x)$  goes to zero sufficiently fast when  $x$  goes to zero and to infinity. At the same time we need the integral

$$\int_0^{\infty} dt \rho(t) t \quad (177)$$

to be finite. We can also check that the integral

$$\int_0^{\infty} dt \rho(t) \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} \quad (178)$$

is finite and nonzero if  $\rho(t)$  is non-negative.<sup>29</sup> To this end let us first note that the numerator  $2p^2 + t - \sqrt{t^2 + 4tp^2}$  is non-negative since  $2p^2 + t \geq \sqrt{t^2 + 4tp^2}$  when  $t, p^2 \geq 0$ . Moreover, if we define

$$\Phi(t, p^2) = \frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} \quad (182)$$

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<sup>29</sup> Our results cannot be easily extended to the general case of a spectral density  $\rho(t)$  that is negative for some values of  $t$ . However, they apply if one can explicitly verify that the integral in Eq. (178) is indeed finite and nonzero. Also note that we cannot simply consider the limit  $p \rightarrow 0$  of the integral (178), since the factor multiplying  $\rho(t)$  vanishes in this limit and we might erroneously conclude that the above expression is equal to zero. Indeed, already in the tree-level case, i.e. for  $\rho(t) = 2\delta(t)$ , one finds that the integral (178) is non-zero and equal to 2. Finally, one could also formally expand the integrand in powers of  $p^2$  leading to

$$\frac{2p^2 + t - \sqrt{t^2 + 4tp^2}}{p^2} = -4 \sum_{n=2}^{\infty} \binom{1/2}{n} \left( \frac{4p^2}{t} \right)^{n-1}, \quad (179)$$

where  $\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$  is the usual binomial coefficient. However, this series is not converging for all values of  $t \in [0, +\infty)$ .

Moreover, for  $n = 2$  we have a term proportional to

$$\int_0^{\infty} dt \frac{\rho(t)}{t} \quad (180)$$

and this integral is divergent. Indeed, by (formally) setting  $p^2 = 0$  in Eq. (152) we find

$$\int_0^{\infty} dt \frac{\rho(t)}{t} = \infty. \quad (181)$$

it is clear that  $2 = \Phi(0, p^2) \geq \Phi(t, p^2) \geq 0$ , since the quantity  $t - \sqrt{t^2 + 4tp^2}$  is negative for  $t > 0$ . This implies

$$\int_0^\infty dt \rho(t) \Phi(t, p^2) < 2 \int_0^\infty dt \rho(t) = 2, \quad (183)$$

where we used again the normalization condition (154). At the same time we can write

$$\int_0^\infty dt \rho(t) \Phi(t, p^2) = \int_0^\infty dt \frac{p^2 \rho(t)}{t + p^2} \frac{(t + p^2) \Phi(t, p^2)}{p^2} > \frac{3}{2} \int_0^\infty dt \frac{p^2 \rho(t)}{t + p^2}, \quad (184)$$

where we use the fact that the function  $(t + p^2)\Phi(t, p^2)/p^2$  is positive and gets its minimum value, equal to  $3/2$ , for  $t/p^2 = 1/2$ . Then, using the definition (153) and the condition  $\sigma(p^2) \geq 0$ , we can write

$$\int_0^\infty dt \rho(t) \Phi(t, p^2) > \frac{3}{2[1 - \sigma(p^2)]} \geq \frac{3}{2}. \quad (185)$$

From the above results we conclude that in Eq. (173) we have two possible IR singularities, i.e. the term  $\mathcal{D}(0)/[1 - \sigma(p^2)]$ , if  $\sigma(0) = 1$ , and the term proportional to  $-\mathcal{D}(p^2) \ln(p^2)$ . In both cases we need to impose the condition  $\mathcal{D}(0) = 0$  in order to avoid the singularity. Thus, we find again that a massive gluon propagator in the  $d = 2$  case is not compatible with the restriction of the functional integration to the first Gribov region.

### B. The 3d Case

In the 3d case we expect no UV divergences when using dimensional regularization<sup>30</sup> and the DSE for the Gribov form-factor is simply

$$\frac{\sigma(p^2)}{g^2 N_c} = \frac{1}{p^2} \int \frac{d^3 q}{(2\pi)^3} p_\lambda \Gamma_{\lambda\nu}(p, q) s_\nu \mathcal{D}(q^2) P_{\mu\nu}(q) \frac{1}{s^2} \frac{1}{1 - \sigma(s^2)} \quad (186)$$

$$= \int_0^\infty dq \frac{q^2}{(2\pi)^3} \mathcal{D}(q^2) \int d\Omega_3 \frac{1 - \cos^2(\phi_1)}{s^2 [1 - \sigma(s^2)]}, \quad (187)$$

where we used the tree-level ghost-gluon vertex  $\Gamma_{\lambda\nu}(p, q) = \delta_{\lambda\nu}$  and  $s^2 = p^2 + q^2 - 2pq \cos(\phi_1)$ . We can now work as in Section IV A 1 and use the results of Appendix B. In this way we obtain the upper bounds

$$\frac{\sigma(p^2)}{g^2 N_c} \leq \frac{I_3(p^2, \infty)}{1 - \sigma(0)}, \quad (188)$$

if  $\sigma(p^2) \leq \sigma(0) < 1$ , and

$$\frac{\sigma(p^2)}{g^2 N_c} \leq \left( \frac{M''}{c} + M \right) I_3(p^2, \infty), \quad (189)$$

if  $\sigma(p^2) \leq \sigma(0) = 1$  with  $\sigma(p^2) \approx 1 - cp^{2\kappa} + \mathcal{O}(p^\tau)$  for small  $p^2$ . In the latter case we also need the conditions  $1 > \kappa$  and  $\tau \geq 4\kappa$ . As we saw in Eq. (B35), under simple assumptions for the gluon propagator  $\mathcal{D}(q^2)$ , the integral  $I_3(p^2, \infty)$  is finite in the IR limit  $p \rightarrow 0$ . Thus, in both cases the upper bound of  $\sigma(p^2)$  is also finite<sup>31</sup> and, in order to have a finite value for  $\sigma(0)$  in the 3d case we do not need to set  $\mathcal{D}(0) = 0$ . This result also applies when an IR-finite ghost-gluon vertex is included in the ghost DSE (186). Let us also note that the scaling solution predicts in the 3d case [14–16] a value  $\kappa \approx 0.4$  for which the condition  $1 > \kappa > 0$  is satisfied.

<sup>30</sup> As shown in Section III B above at one-loop level, the evaluation of the ghost propagator in 3d usually involves Gamma functions with half-integer arguments, which do not generate infinities. Indeed, for nonnegative values of  $n$  with  $n$  integer, one has [97]  $\Gamma(n + 1/2) = \sqrt{\pi} 2^{-n} (2n - 1)!!$  and  $\Gamma(-n + 1/2) = (-2)^n \sqrt{\pi} / (2n - 1)!!$ , where  $n!!$  denotes the double factorial.

<sup>31</sup> Using the fact that  $\sigma(p^2)$  is nonnegative and Eqs. (B31) and (B36), the lower bound

$$\frac{3}{4} I_3(p^2, \infty) \leq I(p^2, 1, 3, \infty) \leq \frac{\sigma(p^2)}{g^2 N_c} \quad (190)$$

clearly applies. However, since  $\sigma(p^2)$  is finite, this bound does not add any relevant information to our analysis. Note that this observation can be made also for the 4d case described in Section IV C.

### C. The 4d Case

In 4d the DSE for  $\sigma(p^2)$  is given by (see for example [94])

$$\sigma(p^2) = 1 - \tilde{Z}_3 + \tilde{Z}_1 g^2 N_c \int_0^\infty dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2 [1 - \sigma(s^2)]}, \quad (191)$$

where  $\tilde{Z}_3$  and  $\tilde{Z}_1$  are the renormalization constants for the ghost propagator and the ghost-gluon vertex respectively<sup>32</sup> and  $s^2 = p^2 + q^2 - 2pq \cos(\phi_1)$ . In order to eliminate these constants from the expression for  $\sigma(p^2)$  we can subtract<sup>33</sup> the same equation for some fixed value  $p^2 = \mu^2$  and set  $\tilde{Z}_1 = 1$ , using the non-renormalization of the ghost-gluon vertex in Landau gauge [103]. This gives

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &= \frac{\sigma(\mu^2)}{g^2 N_c} + \int_0^\infty dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2 [1 - \sigma(s^2)]} \\ &\quad - \int_0^\infty dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{t^2 [1 - \sigma(t^2)]}, \end{aligned} \quad (192)$$

with  $t^2 = \mu^2 + q^2 - 2\mu q \cos(\phi_1)$ . Clearly, considering  $\mathcal{D}(q^2) \sim 1/q^2$  at large momenta, the UV (logarithmic) divergence of the two integrals cancels out. In order to derive upper bounds for the above expression without spoiling the cancellation of UV divergences and since we are interested in the IR limit  $p \rightarrow 0$ , we can consider a momentum scale  $\ell$  sufficiently large and write

$$\begin{aligned} \frac{\sigma(p^2)}{g^2 N_c} &= \frac{\sigma(\mu^2)}{g^2 N_c} + \int_0^\ell dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2 [1 - \sigma(s^2)]} \\ &\quad + \int_\ell^\infty dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2 [1 - \sigma(s^2)]} - \int_0^\infty dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{t^2 [1 - \sigma(t^2)]}. \end{aligned} \quad (193)$$

For small momenta  $p$  only the first integral on the r.h.s. of the above equation can produce an IR singularity. Following the analysis presented in the 3d case above we can then write

$$\int_0^\ell dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2 [1 - \sigma(s^2)]} \leq \frac{I_4(p^2, \ell)}{1 - \sigma(0)}, \quad (194)$$

if  $\sigma(p^2) \leq \sigma(0) < 1$ , and

$$\int_0^\ell dq \frac{q^3}{(2\pi)^4} \mathcal{D}(q^2) \int d\Omega_4 \frac{1 - \cos^2(\phi_1)}{s^2 [1 - \sigma(s^2)]} \leq \left( \frac{M''}{c} + M \right) I_4(p^2, \ell), \quad (195)$$

if  $\sigma(p^2) = 1 - cp^{2\kappa} + \mathcal{O}(p^\tau)$  with  $\tau \geq 4\kappa$  and  $3/2 > \kappa$ . Again, thanks to the result (B35), both upper bounds are finite in the IR limit  $p \rightarrow 0$  also for  $\mathcal{D}(0) > 0$ .

An alternative proof can be given by working directly with Eq. (191) and using dimensional regularization, i.e. considering a dimension  $d = 4 - \varepsilon$ . In this case we can write

$$\sigma(p^2) = 1 - \tilde{Z}_3 + \tilde{Z}_1 \sigma_d(p^2) \quad (196)$$

with<sup>34</sup>

$$\frac{\sigma_d(p^2)}{g^2 N_c} = \int_0^\infty dq \frac{q^{d-1}}{(2\pi)^d} \mathcal{D}(q^2) \int d\Omega_d \frac{1 - \cos^2(\phi_1)}{s^2 [1 - \sigma_d(s^2)]}. \quad (197)$$

<sup>32</sup> Note that we are again considering a tree-level ghost-gluon vertex  $\Gamma_{\lambda\nu}(p, q) = \delta_{\lambda\nu}$ .

<sup>33</sup> Again, this corresponds to a MOM scheme with the condition  $\mathcal{G}(\mu^2) = 1/\mu^2$ .

<sup>34</sup> Of course, with  $d = 4 - \varepsilon$  and  $\varepsilon > 0$ , the integral in Eq. (197) is no longer dimensionless. To keep the dimensionality correct we should, as always, scale out a dimensional factor  $m^{4-d}$  where  $m$  is a mass scale, which could then be combined with the coupling constant  $g^2$ , making  $\sigma(p^2)$  dimensionless also for  $\varepsilon > 0$ . This is important when evaluating the  $\varepsilon$  expansion in order to single out UV divergencies. Since here we are mainly interested in the IR behavior of  $\sigma(p^2)$ , we do not keep track explicitly of all the terms depending on  $\varepsilon$  and we simply consider the coupling  $g^2$  dimensionful.

Then, if we can show that no IR singularities occur for  $2 < d \leq 4$ , the UV infinity that appears for  $d \rightarrow 4$  is taken care of by the renormalization factors. In order to show that  $\sigma_d(p^2)$  is IR finite we can work as done above and write

$$\frac{\sigma_d(p^2)}{g^2 N_c} \leq \frac{I_4(p^2, \infty)}{1 - \sigma(0)}, \quad (198)$$

or

$$\frac{\sigma_d(p^2)}{g^2 N_c} \leq \left( \frac{M''}{c} + M \right) I_4(p^2, \ell), \quad (199)$$

depending on the value of  $\sigma_d(0)$ . In the latter case we considered again the IR expansion  $\sigma_d(p^2) \approx 1 - cp^{2\kappa} + \mathcal{O}(p^\tau)$  and the conditions  $\tau \geq 4\kappa$  and  $3/2 > \kappa$ . We conclude that also in the  $4d$  case,  $\sigma(0)$  is finite if  $\mathcal{D}(0)$  is also finite (but not necessarily null).

Let us remark that the IR exponent usually obtained in the scaling solution [14–16] is  $\kappa \approx 0.6$  in the  $4d$  case, i.e. the condition  $\kappa < 3/2$  is satisfied. Also note that when  $(d-1)/2 \leq \kappa$  the hypergeometric function  ${}_2F_1(1+\kappa, 1+\kappa-d/2; 1+d/2; z)$  is not convergent at  $z=1$  and we cannot use the above proofs in order to derive properties of the Gribov form-factor. However, these large values of  $\kappa$  imply for the ghost propagator  $\mathcal{G}(p^2)$  a very strong IR enhancement with a behavior at least as singular as  $1/k^5$  in  $4d$  and at least as singular as  $1/k^4$  for  $d=3$ .

## V. CONCLUSION

Summarizing, in this manuscript we have considered general properties of the Landau-gauge Gribov ghost form-factor  $\sigma(p^2)$  for  $SU(N_c)$  Euclidean Yang-Mills theories in  $d \geq 2$  space-time dimensions. This form-factor is in a one-to-one correspondence with the ghost propagator  $\mathcal{G}(p^2)$  via Eq. (2). Also,  $\sigma(p^2)$  is bounded by 1 if the no-pole condition (3) is imposed, i.e. if one restricts the functional integration to the first Gribov region  $\Omega$ . The main result of this work is an exact proof of qualitatively different behavior of  $\sigma(p^2)$  for  $d=3, 4$  with respect to  $d=2$ . In particular, for  $d=2$ , the gluon propagator  $\mathcal{D}(p^2)$  needs to vanish at zero momentum in order to avoid in  $\sigma(p^2)$  an IR singularity proportional to  $-\mathcal{D}(0) \ln(p^2)$ . On the contrary, for  $d=3$  and  $4$ , an IR-finite ghost form-factor  $\sigma(p^2)$  is obtained also when  $\mathcal{D}(0) > 0$ . These results were proven, in Section II, using perturbation theory at one loop and, in Section IV, by considering the DSE for the ghost propagator. Let us stress again that in DSE studies of correlation functions in minimal Landau gauge, besides using the no-pole condition, a specific boundary condition is usually imposed on the Gribov ghost form-factor at zero momentum. Here, instead, we have tried to prove general properties of the Gribov ghost form-factor  $\sigma(p^2)$  when the restriction to the first Gribov horizon is considered.

At the same time, in Section III, we have presented closed analytic expressions for the Gribov form-factor  $\sigma(p^2)$  at one loop, considering for the gluon propagator linear combinations of Yukawa-like propagators (with real and/or complex-conjugate poles). These functional forms, briefly described in Eqs. (53)–(55), were recently used to fit lattice data of the gluon propagator in the  $SU(2)$  case [59, 60]. The expressions obtained for  $\sigma(p^2)$  confirm the results presented in Section II. These expressions also show that, for the ghost propagator  $\mathcal{G}(p^2)$ , there is a one-parameter family of behaviors [21, 34] labelled by the coupling constant  $g^2$ , when it is considered as a free parameter. The no-pole condition  $\sigma(0) \leq 1$  then implies  $g^2 \leq g_c^2$ , where  $g_c^2$  is a “critical” value. For  $g^2$  smaller than  $g_c^2$  one has  $\sigma(0) < 1$  and the ghost propagator is a massive one. On the contrary, at the “critical” value  $g_c^2$ , i.e. for  $\sigma(0) = 1$ , one finds an IR-enhanced ghost propagator. As stressed in the Introduction, the physical value of the coupling is expected to select the actual value of  $\sigma(0)$ . Present results [21, 34] give  $\sigma(0) < 1$  in the four-dimensional  $SU(3)$  case.

Our findings imply that a massive gluon propagator cannot be obtained in the two-dimensional case, in disagreement with some of the results presented in Ref. [16] (see their Table 2).<sup>35</sup> A possible massive behavior for the gluon propagator in the  $2d$  case was also explicitly conjectured in Ref. [65] as a Gribov-copy effect. However, since our  $2d$

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<sup>35</sup> A massive solution in the  $2d$  case was also obtained in Ref. [14]. On the other hand, the author of [14] stressed that a full understanding of the  $2d$  case would require a more detailed investigation.

result is valid for any Gribov copy inside the first Gribov region, we have shown that, at least for the  $2d$  gluon propagator  $\mathcal{D}(p^2)$  in the minimal Landau gauge, Gribov-copy effects do not alter our conclusion for the value of the gluon propagator at zero momentum, i.e.  $\mathcal{D}(0)$  must vanish. This observation also represents an explicit counterexample to the identification of the one-parameter family of solutions for the gluon and ghost DSEs [22, 31] with different Gribov copies, as suggested in [65, 67, 68].

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### Appendix A: Angular Integration in the $d$ -Dimensional Case

Following Appendix B in Ref. [15] one can easily perform, using hyperspherical coordinates, the angular integrations necessary for our calculations. To this end let us recall that, in  $d$  dimensions, one has the following relations between Cartesian coordinates  $x_i$  (with  $i = 1, 2, \dots, d$ ) and hyperspherical coordinates  $r, \phi_j$  (with  $j = 1, 2, \dots, d-1$ ):

$$\begin{aligned} x_1 &= r \cos(\phi_1) , \\ x_2 &= r \sin(\phi_1) \cos(\phi_2) , \\ x_3 &= r \sin(\phi_1) \sin(\phi_2) \cos(\phi_3) , \\ &\dots \\ x_{d-1} &= r \sin(\phi_1) \sin(\phi_2) \dots \sin(\phi_{d-2}) \cos(\phi_{d-1}) , \\ x_d &= r \sin(\phi_1) \sin(\phi_2) \dots \sin(\phi_{d-2}) \sin(\phi_{d-1}) . \end{aligned} \quad (\text{A1})$$

The hyperspherical coordinates take values, respectively,  $r \in [0, \infty)$ ,  $\phi_i \in [0, \pi]$  for  $i = 1, 2, \dots, d-2$  and  $\phi_{d-1} \in [0, 2\pi)$ . At the same time, the volume measure is given by

$$dV = r^{d-1} dr d\Omega_d \quad (\text{A2})$$

with

$$d\Omega_d = \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) \dots \sin(\phi_{d-2}) d\phi_1 d\phi_2 \dots d\phi_{d-2} d\phi_{d-1} . \quad (\text{A3})$$

Note that, in the usual three-dimensional case, this notation correspond to  $x_1 = z$ ,  $x_2 = x$  and  $x_3 = y$ .

Here we want to evaluate the integral

$$\int d\Omega_d f(\phi_1) . \quad (\text{A4})$$

If we indicate with  $\Omega_d$  the well-known result

$$\Omega_d = \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} , \quad (\text{A5})$$

where  $\Gamma(x)$  is the Gamma function, then we have

$$\int f(\phi_1) d\Omega_d = \frac{\Omega_d}{\int_0^\pi \sin^{d-2}(\phi_1) d\phi_1} \int_0^\pi \sin^{d-2}(\phi_1) f(\phi_1) d\phi_1 , \quad (\text{A6})$$

where all other angular integrations have already been evaluated. The integral in the denominator can be written as

$$\int_0^\pi \sin^{d-2}(\phi_1) d\phi_1 = \int_{-1}^1 (1-z^2)^{(d-3)/2} dz = \int_0^1 t^{-1/2} (1-t)^{(d-3)/2} dt = B\left(\frac{d-1}{2}, \frac{1}{2}\right) , \quad (\text{A7})$$

where  $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$  is the Beta function.

In our calculations we consider two different functions  $f(\phi_1)$ , i.e.

$$f(\phi_1) = 1 - \cos^2(\phi_1) \quad (\text{A8})$$

and

$$f(\phi_1) = \frac{1 - \cos^2(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu} . \quad (\text{A9})$$

In the first case the integration gives

$$\int [1 - \cos^2(\phi_1)] d\Omega_d = \frac{\Omega_d}{B(\frac{d-1}{2}, \frac{1}{2})} \int_0^\pi \sin^{d-2}(\phi_1) [1 - \cos^2(\phi_1)] d\phi_1 \quad (\text{A10})$$

and the integral in the numerator is

$$\int_{-1}^1 (1 - z^2)^{(d-1)/2} dz = \int_0^1 t^{-1/2} (1 - t)^{(d-1)/2} dt = B\left(\frac{d+1}{2}, \frac{1}{2}\right) . \quad (\text{A11})$$

Collecting these results we find

$$\int [1 - \cos^2(\phi_1)] d\Omega_d = \Omega_d \frac{B(\frac{d+1}{2}, \frac{1}{2})}{B(\frac{d-1}{2}, \frac{1}{2})} = \Omega_d \frac{d-1}{d} , \quad (\text{A12})$$

where we used  $x\Gamma(x) = \Gamma(x+1)$ . In the second case, i.e. when considering the integral

$$\int \frac{1 - \cos^2(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu} d\Omega_d , \quad (\text{A13})$$

we have

$$\frac{\Omega_d}{B(\frac{d-1}{2}, \frac{1}{2})} \int_0^\pi \frac{\sin^d(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu} d\phi_1 . \quad (\text{A14})$$

We can now use the result (see for example formula 3.665.2 in [97])

$$\int_0^\pi \frac{\sin^{2\mu-1}(\theta)}{[1 + a^2 \pm 2a \cos(\theta)]^\nu} d\theta = B(\mu, 1/2) {}_2F_1(\nu, \nu - \mu + 1/2; \mu + 1/2; a^2) , \quad (\text{A15})$$

which is valid for  $|a| < 1$  and  $\text{Re}(\mu) > 0$ . Here  ${}_2F_1(a, b; c; z)$  is the Gauss hypergeometric function (see Appendix B). Then, we find

$$\int \frac{1 - \cos^2(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu} d\Omega_d = \frac{\Omega_d}{B(\frac{d-1}{2}, \frac{1}{2})} \frac{1}{q^2} B\left(\frac{d+1}{2}, \frac{1}{2}\right) {}_2F_1(\nu, \nu - d/2; 1 + d/2; p^2/q^2) \quad (\text{A16})$$

$$= \frac{\Omega_d}{q^2} \frac{d-1}{d} {}_2F_1(\nu, \nu - d/2; 1 + d/2; p^2/q^2) \quad (\text{A17})$$

if  $p^2 < q^2$  and

$$\int \frac{1 - \cos^2(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu} d\Omega_d = \frac{\Omega_d}{p^2} \frac{d-1}{d} {}_2F_1(\nu, \nu - d/2; 1 + d/2; q^2/p^2) \quad (\text{A18})$$

if  $q^2 < p^2$ .

## Appendix B: Properties of the Gauss Hypergeometric Function

Let us recall that the Gauss hypergeometric function  ${}_2F_1(a, b; c; z)$  is defined [97] for  $|z| < 1$  by the series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{a b}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2} \dots, \quad (\text{B1})$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (\text{B2})$$

is a so-called Pochhammer symbol. This series is converging for  $c \neq 0, -1, -2, \dots$ . It is also converging for  $|z| = 1$  if  $\Re(c - a - b) > 0$ , where  $\Re$  indicates the real part. When this condition is satisfied one can use, for  $z = 1$ , the result (see formula 9.122.1 in Ref. [97])

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - b) \Gamma(c - a)}. \quad (\text{B3})$$

In Eqs. (A17) and (A18) the hypergeometric function appears with  $c = 1 + d/2$ . Therefore, the corresponding series is converging inside the unit circle for any dimension  $d > 0$ . At the same time we have  $c - a - b = d + 1 - 2\nu$  and  ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$  is finite on the unit circle for  $(d + 1)/2 > \nu$ . Also, using the relation (see equation 9.131.1 in Ref. [97])

$${}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z) \quad (\text{B4})$$

we can write

$${}_2F_1(\nu, \nu - d/2; 1 + d/2; z) = (1 - z)^{d+1-2\nu} {}_2F_1(1 + d/2 - \nu, 1 + d - \nu; 1 + d/2; z) \quad (\text{B5})$$

and for  $(d+1)/2 - \nu > 0$  we have that  $1 + d/2 - \nu, 1 + d - \nu > 0$ , i.e. the hypergeometric function  ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$  is clearly positive for  $z \in [0, 1]$ .

In the case  $\nu = 1$  the above results simplify. In particular, we have convergence of the series on the unit circle for any dimension  $d > 1$ . Then Eq. (B3) yields

$${}_2F_1(1, 1 - d/2; 1 + d/2; 1) = \frac{\Gamma(1 + d/2) \Gamma(d - 1)}{\Gamma(d) \Gamma(d/2)} = \frac{d/2}{d - 1}, \quad (\text{B6})$$

which is positive. Note that for  $d = 2$  the above result gives  ${}_2F_1(1, 0; 2; 1) = 1$ . One can actually check that, when  $d = 2$ , the function  ${}_2F_1(1, 1 - d/2; 1 + d/2; z) = {}_2F_1(1, 0; 2; z)$  is simply equal to 1 for any value  $|z| \leq 1$ . Indeed, from the series representation (B1) it is obvious that  ${}_2F_1(a, b; c; z) = 1$  for  $b = 0$  (and/or for  $a = 0$ ). The same result can be obtained by considering Eq. (B4) and the relation (see equation 9.121.1 in Ref. [97])

$${}_2F_1(n, c; c; z) = (1 - z)^{-n}, \quad (\text{B7})$$

yielding

$${}_2F_1(a, 0; c; z) = (1 - z)^{c-a} {}_2F_1(c - a, c; c; z) = (1 - z)^{c-a} (1 - z)^{a-c} = 1. \quad (\text{B8})$$

The hypergeometric function  ${}_2F_1(1, 1 - d/2; 1 + d/2; z)$  is actually very simple for any even dimension  $d = 2n$ , with  $n \geq 1$ . Indeed, in this case we are considering the function  ${}_2F_1(1, 1 - n; 1 + n; z)$  and the coefficient  $b$  is either zero or negative. This implies that the series (B1) is actually a polynomial in  $z$ . For example, for  $d = 4$  we have the very simple expression

$${}_2F_1(1, -1; 3; z) = 1 - \frac{z}{3}. \quad (\text{B9})$$



Note that it is also possible to find a closed form for the hypergeometric function  ${}_2F_1(1, 1 - d/2; 1 + d/2; z)$  in the  $3d$  case. Indeed, by considering the relations (see equations 9.137.8 and 9.137.1 in Ref. [97])

$$0 = {}_2F_1(a, b; c; z) c + {}_2F_1(a + 1, b; c + 1; z) (b - c) - {}_2F_1(a + 1, b + 1; c + 1; z) b (1 - z) \quad (\text{B10})$$

$$0 = {}_2F_1(a, b; c; z) c [c - 1 - (2c - a - b - 1) z] + {}_2F_1(a, b; c + 1; z) (c - a) (c - b) z + {}_2F_1(a, b; c - 1; z) c (c - 1) (z - 1) \quad (\text{B11})$$

and (see equation 9.131.1 in [97])

$${}_2F_1(a, b; c; z) = (1 - z)^{-b} {}_2F_1\left(b, c - a; c; \frac{z}{z - 1}\right) \quad (\text{B12})$$

we can write

$${}_2F_1(1, -1/2; 5/2; z) = \frac{3}{4} {}_2F_1(0, -1/2; 3/2; z) + \frac{1 - z}{4} {}_2F_1(1, 1/2; 5/2; z) \quad (\text{B13})$$

$$= \frac{3}{4} + \frac{1 - z}{4} \frac{3(1 - z)}{2z} [{}_2F_1(1, 1/2; 1/2; z) - {}_2F_1(1, 1/2; 3/2; z)] \quad (\text{B14})$$

$$= \frac{3}{4} + \frac{3(1 - z)}{8z} [1 - (1 - z) {}_2F_1(1, 1/2; 3/2; z)] \quad (\text{B15})$$

$$= \frac{3}{4} + \frac{3(1 - z)}{8z} \left[ 1 - \sqrt{1 - z} {}_2F_1\left(1/2, 1/2; 3/2; \frac{z}{z - 1}\right) \right] \quad (\text{B16})$$

$$= \frac{3}{4} + \frac{3(1 - z)}{8z} \left[ 1 - \frac{1 - z}{\sqrt{z}} \operatorname{arcsinh}\left(\sqrt{\frac{z}{1 - z}}\right) \right], \quad (\text{B17})$$

where we have also used Eq. (B7), the relations  ${}_2F_1(0, b; c; z) = 0$  and (see 9.121.27 in [97])

$${}_2F_1(1/2, 1/2; 3/2; -z^2) = \frac{\operatorname{arcsinh} z}{z}. \quad (\text{B18})$$

From the expression (B17) it is easy to check that  ${}_2F_1(1, -1/2; 5/2; 1) = 3/4$  and that  ${}_2F_1(1, -1/2; 5/2; 0) = 1$ , as expected.

Using the above series (B1) one can verify that the derivative of  ${}_2F_1(a, b; c; z)$ , with respect to the variable  $z$ , is given by

$$\frac{\partial}{\partial z} {}_2F_1(a, b; c; z) = \frac{ab}{c} {}_2F_1(a + 1, b + 1; c + 1; z). \quad (\text{B19})$$

Thus, in the case of interest for us, we have

$$\frac{\partial}{\partial z} {}_2F_1(\nu, \nu - d/2; 1 + d/2; z) = \frac{\nu(\nu - d/2)}{1 + d/2} {}_2F_1(\nu + 1, \nu + 1 - d/2; 2 + d/2; z). \quad (\text{B20})$$

These results can be written as

$$\frac{\partial}{\partial z} {}_2F_1(\nu, \nu - d/2; 1 + d/2; z) = \frac{\nu(\nu - d/2)}{1 + d/2} (1 - z)^{d - 2\nu} {}_2F_1(1 + d/2 - \nu, 1 + d - \nu; 2 + d/2; z), \quad (\text{B21})$$

where we made use of Eq. (B4). When the hypergeometric function  ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$  is finite in the unit circle, i.e. for  $(d + 1)/2 > \nu$ , it is clear that this derivative is positive for  $\nu > d/2$  and  ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$  attains its maximum value at  $z = 1$ , equal to

$${}_2F_1(\nu, \nu - d/2; 1 + d/2; 1) = \frac{\Gamma(1 + d/2) \Gamma(1 + d - 2\nu)}{\Gamma(1 + d - \nu) \Gamma(1 + d/2 - \nu)}, \quad (\text{B22})$$

and its minimum value, equal to 1, at  $z = 0$ . On the contrary, the same derivative is negative when  $\nu < d/2$ . In this case  ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$  is largest at  $z = 0$ , with  ${}_2F_1(\nu, \nu - d/2; 1 + d/2; 0) = 1$ , and smallest at  $z = 1$  with a value given by Eq. (B22). Finally, for  $\nu = d/2$  the derivative is null and the hypergeometric function  ${}_2F_1(\nu, \nu - d/2; 1 + d/2; z)$  is equal to 1 for all values of  $z \in [0, 1]$ . Thus, for  $(d + 1)/2 > \nu$  we can always write  $M' < {}_2F_1(\nu, \nu - d/2; 1 + d/2; z) < M''$ , for some positive constants  $M'$  and  $M''$  and with  $z$  taking values in the interval  $[0, 1]$ .

For  $\nu = 1$  these results again simplify, yielding

$$\frac{\partial}{\partial z} {}_2F_1(1, 1 - d/2; 1 + d/2; z) = \frac{2 - d}{2 + d} {}_2F_1(2, 2 - d/2; 2 + d/2; z). \quad (\text{B23})$$

As expected, for  $d = 2$  this derivative is zero since  ${}_2F_1(1, 0; 2; z) = 1$ . The result above also simplifies for  $d = 4$ , for which we find on the right-hand side of Eq. (B23) the value  $-{}_2F_1(2, 0; 4; z)/3 = -1/3$ , as already known from Eq. (B9). At the same time, for  $d = 3$  we can write

$$\frac{\partial}{\partial z} {}_2F_1(1, -1/2; 5/2; z) = -\frac{1}{5} {}_2F_1(2, 1/2; 7/2; z) = -\frac{1}{5} \left( 1 + \frac{2z}{7} + \frac{z^2}{14} \dots \right) \quad (\text{B24})$$

and the derivative is clearly negative for any value  $z \geq 0$ . The same result can actually be proven for any dimension  $d$  larger than 2. Indeed, the hypergeometric function  ${}_2F_1(2, 2 - d/2; 2 + d/2; z)$  is finite in the unit circle for  $d > 2$  and using Eq. (B4) we can easily verify that it is also positive for  $z \in [0, 1]$ . Thus, the derivative in Eq. (B23) is negative for  $d > 2$  (and  $z \in [0, 1]$ ). As a consequence, under the same hypotheses, we have that the hypergeometric function  ${}_2F_1(1, 1 - d/2; 1 + d/2; z)$  has its maximum value, equal to 1, at  $z = 0$ , and its minimum value, equal to  $d/(2(d - 1))$ , at  $z = 1$  [see Eq. (B6)].

Using the definition (B2) of the Pochhammer symbol  $(a)_n$  it is also easy to verify that

$$(a)_n = a(a + 1)(a + 2) \dots (a + n - 1) \quad (\text{B25})$$

which implies

$$n(a)_n = a[(a + 1)_n - (a)_n]. \quad (\text{B26})$$

Then, using Eq. (B1), one can prove the relation [104]

$$z \frac{\partial}{\partial z} {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} n \frac{z^n}{n!} \quad (\text{B27})$$

$$= a \left[ \sum_{n=0}^{\infty} \frac{(a + 1)_n (b)_n}{(c)_n} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \right] \quad (\text{B28})$$

$$= a [{}_2F_1(a + 1, b; c; z) - {}_2F_1(a, b; c; z)]. \quad (\text{B29})$$

In particular, we can write

$$z \frac{\partial}{\partial z} {}_2F_1(1, 1 - d/2; 1 + d/2; z) + {}_2F_1(1, 1 - d/2; 1 + d/2; z) = {}_2F_1(2, 1 - d/2; 1 + d/2; z). \quad (\text{B30})$$

Note that the r.h.s. in the above relation is finite in the unit circle for  $d > 2$  and, using again Eq. (B4), we can verify that it is also positive for  $z \in [0, 1]$ .

The above results, together with the Eqs. (A17) and (A18), allow to write a lower and an upper bound for the integral

$$I(p^2, \nu, d, \ell) = \int_0^\ell dq \frac{q^{d-1}}{(2\pi)^d} \mathcal{D}(q^2) \int d\Omega_d \frac{1 - \cos^2(\phi_1)}{[p^2 + q^2 - 2pq \cos(\phi_1)]^\nu}. \quad (\text{B31})$$

Indeed, after considering the angular integration, we have (for  $\ell > p$ )

$$I(p^2, \nu, d, \ell) = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \left[ \int_0^p dq q^{d-1} \frac{\mathcal{D}(q^2)}{p^2} {}_2F_1(\nu, \nu - d/2; 1 + d/2; q^2/p^2) + \int_p^\ell dq q^{d-3} \mathcal{D}(q^2) {}_2F_1(\nu, \nu - d/2; 1 + d/2; p^2/q^2) \right]. \quad (\text{B32})$$

Then, for  $1 + d - 2\nu > 0$  we obtain

$$M' I_d(p^2, \ell) \leq I(p^2, \nu, d, \ell) \leq M'' I_d(p^2, \ell) \quad (\text{B33})$$

with<sup>36</sup>

$$I_d(p^2, \ell) = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \left[ \int_0^p dq q^{d-1} \frac{\mathcal{D}(q^2)}{p^2} + \int_p^\ell dq q^{d-3} \mathcal{D}(q^2) \right]. \quad (\text{B34})$$

Also, in the limit  $p \rightarrow 0$ , we find (for  $d > 1$ )

$$\lim_{p \rightarrow 0} I_d(p^2, \ell) = \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \left[ \lim_{p \rightarrow 0} \frac{p^{d-2} \mathcal{D}(p^2)}{2} + \int_0^\ell dq q^{d-3} \mathcal{D}(q^2) \right], \quad (\text{B35})$$

where we used the trapezoidal rule. Clearly, for  $\mathcal{D}(0) > 0$ , the first term is IR-finite if  $d \geq 2$  while the second term is finite for  $d > 2$ . Finally, note that for  $\nu = 1$  the condition  $1 + d - 2\nu > 0$  simplifies to  $d > 1$ . At the same time, for  $d \geq 2$  the inequalities (B33) become

$$\frac{d}{2(d-1)} I_d(p^2, \ell) \leq I(p^2, \nu, d, \ell) \leq I_d(p^2, \ell) \quad (\text{B36})$$

and for  $d = 2$  we have  $I(p^2, \nu, 2, \ell) = I_2(p^2, \ell)$ .

### Appendix C: Hypotheses on the $2d$ Gluon Propagator

In Section II B we have proven, using two different approaches, that in the  $2d$  case one needs to set  $\mathcal{D}(0) = 0$  in order to have  $\sigma(p^2) < +\infty$ . The assumptions made for the gluon propagator were rather general. Indeed, for the first proof one needs, for small momenta  $p^2$ , an expansion of the gluon propagator of the type  $\mathcal{D}(p^2) \approx \mathcal{D}(0) + B p^{2\eta} + C p^{2\xi}$ , with  $\xi > \eta > 0$  and  $\mathcal{D}(0), B$  and  $C$  finite. At the same time, for large momenta  $p^2$ , we required

$$\lim_{p^2 \rightarrow \infty} \mathcal{D}(p^2) = \lim_{p^2 \rightarrow \infty} \frac{\hat{D}(p^2)}{p^2} = 0. \quad (\text{C1})$$

Let us recall that we are indicating with  $\hat{D}(p^2)$  a primitive of  $\mathcal{D}(p^2)$  and that  $\mathcal{D}'(p^2)$  is the first derivative with respect to the variable  $p^2$ . In Section II B above we considered for  $\mathcal{D}(p^2)$  a large  $p^2$  behavior of the type  $1/p^2$ . However, it is clear that a weaker condition can also be used. Indeed, the behavior  $\mathcal{D}(p^2) \sim 1/p^{2\epsilon}$  with  $1 > \epsilon > 0$  also allows to satisfy the above conditions. In order to check this, one should recall that  $\mathcal{D}(p^2) \sim 1/p^{2\epsilon}$  implies<sup>37</sup>  $\hat{D}(p^2) \sim p^{2-2\epsilon} + \text{constant}$  and these two asymptotic behaviors yield the limits in Eq. (C1). Under the same hypothesis we can also verify that the integral [see Eq. (26)]

$$\int_0^\infty dx \ln(x^\eta + M) \frac{\eta M [\mathcal{D}(x) - \mathcal{D}(0)] - x (x^\eta + M) \mathcal{D}'(x)}{x^{1+\eta}} \quad (\text{C2})$$

<sup>36</sup> Note that, for a gluon propagator  $\mathcal{D}(q^2)$  with a behavior  $1/q^2$  at large momenta, the second integral in Eq. (B34) is UV divergent if  $d \geq 4$  and  $\ell = \infty$ .

<sup>37</sup> Let us recall that, while this (Abelian theorem) is a correct statement, the converse (also called Tauberian theorem), i.e.  $\hat{D}(p^2) \sim p^{2-2\epsilon}$  implies  $\mathcal{D}(p^2) \sim 1/p^{2\epsilon}$ , is not always true (see for example Ref. [105] and Section 7.3 in Ref. [106]). This is why the so-called de l'Hôpital's rule does not always apply (see also footnote 39).

is finite. To this end, let us first consider the integral

$$\int_0^{p^2} dx \ln(x^\eta + M) \frac{\eta M [\mathcal{D}(x) - \mathcal{D}(0)] - x (x^\eta + M) \mathcal{D}'(x)}{x^{1+\eta}} \quad (\text{C3})$$

with  $0 < p$  and  $p$  small. In the limit  $x \rightarrow 0$  the integrand behaves as

$$\ln(M) [MC(\eta - \xi) x^{\xi - \eta - 1} - B \eta x^{\eta - 1}] \quad (\text{C4})$$

and no singularity arises in the integration from  $x = 0$  to  $x = p^2$  (if  $\xi > \eta > 0$ ). At the same time, for large  $x$  the same integrand behaves as

$$\ln(x) \left[ \frac{\eta M \mathcal{D}(0)}{x^{1+\eta}} - \mathcal{D}'(x) \right]. \quad (\text{C5})$$

Thus, for sufficiently large  $\ell^2$ , the integral

$$\int_{\ell^2}^{\infty} dx \ln(x^\eta + M) \frac{\eta M [\mathcal{D}(x) - \mathcal{D}(0)] - x (x^\eta + M) \mathcal{D}'(x)}{x^{1+\eta}} \quad (\text{C6})$$

can be approximated by

$$I_a(\ell^2) = \int_{\ell^2}^{\infty} dx \ln(x) \left[ \frac{\eta M \mathcal{D}(0)}{x^{1+\eta}} - \mathcal{D}'(x) \right]. \quad (\text{C7})$$

After integrating by parts<sup>38</sup> we then find

$$I_a(\ell^2) = -\ln(x) \left[ \frac{M \mathcal{D}(0)}{x^\eta} + \mathcal{D}(x) \right] \Big|_{\ell^2}^{\infty} + \int_{\ell^2}^{\infty} dx \left[ \frac{M \mathcal{D}(0)}{x^{1+\eta}} + \frac{\mathcal{D}(x)}{x} \right], \quad (\text{C8})$$

which is clearly finite under the assumptions made for the gluon propagator  $\mathcal{D}(p^2)$ . Finally, the remaining term

$$\int_{p^2}^{\ell^2} dx \ln(x^\eta + M) \frac{\eta M [\mathcal{D}(x) - \mathcal{D}(0)] - x (x^\eta + M) \mathcal{D}'(x)}{x^{1+\eta}}, \quad (\text{C9})$$

with  $p \ll \ell$ , can be easily bounded by making the assumption that neither  $\mathcal{D}(x)$  nor  $\mathcal{D}'(x)$  display a singularity for  $x \in [p^2, \ell^2]$ . Thus, we can conclude that the integral (C2) is indeed finite. We further notice that  $I_a(\ell^2)$  is null in the limit  $\ell^2 \rightarrow \infty$ . Going back to the first proof in Section II B, one can verify that the above conditions allow to show that the Gribov form-factor  $\sigma(p^2)$  goes to zero for  $p^2$  going to infinity and that the term  $-\mathcal{D}(0) \ln(p^2)$  is the only singularity of  $\sigma(p^2)$  in the IR limit  $p^2 \rightarrow 0$ .

The situation is, of course, very similar in the second proof. In this case we considered a finite value for  $\mathcal{D}(0)$  and the limit

$$\lim_{p^2 \rightarrow 0} [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2) = 0. \quad (\text{C10})$$

We also imposed, for large momenta  $p^2$ , the limits

$$\lim_{p^2 \rightarrow \infty} [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2) = \lim_{p^2 \rightarrow \infty} \ln(p^2) \mathcal{D}(p^2) = \lim_{p^2 \rightarrow \infty} \frac{\hat{D}(p^2)}{p^2} = 0. \quad (\text{C11})$$

Clearly, any gluon propagator with an IR behavior of the type  $\mathcal{D}(p^2) \approx \mathcal{D}(0) + B p^{2\eta}$ , with  $\eta > 0$  and with  $\mathcal{D}(0)$  and  $B$  finite satisfies the limit (C10). Also, if we make the hypothesis  $\mathcal{D}'(p^2) \sim 1/p^{2+2\epsilon}$ , with  $1 > \epsilon > 0$  for large values of

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<sup>38</sup> Of course, one could also make hypotheses on the first derivative  $\mathcal{D}'(x)$  and avoid the partial integration.

$p^2$  we have, in the same limit,<sup>39</sup>  $\mathcal{D}(p^2) \sim 1/p^{2\epsilon}$  and  $\hat{D}(p^2) \sim p^{2-2\epsilon}$  and one can easily prove the results in Eq. (C11). As a consequence, we can also verify that the integral [see Eq. (33)]

$$I_b(p^2) = \int_{p^2}^{\infty} dx [x \ln(x) - x] \mathcal{D}''(x) \quad (\text{C13})$$

is finite for any  $p^2 \geq 0$ . Indeed, we can integrate by parts<sup>40</sup> obtaining

$$I_b(p^2) = [x \ln(x) - x] \mathcal{D}'(x) \Big|_{p^2}^{\infty} - \int_{p^2}^{\infty} dx \ln(x) \mathcal{D}'(x) \quad (\text{C14})$$

$$= [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2) - \int_{p^2}^{\infty} dx \ln(x) \mathcal{D}'(x). \quad (\text{C15})$$

Note that the integral on the r.h.s. of Eq. (C15) also appears in the second term of Eq. (C7). Thus, using Eqs. (C8) and (C11) we have that, for large  $p^2$ , the integral  $I_b(p^2)$  is finite and  $\lim_{p^2 \rightarrow \infty} I_b(p^2) = 0$ . At the same time, for  $p^2$  going to zero we have  $\mathcal{D}''(p^2) \sim p^{2\eta-4}$  and the integrand in Eq. (C13) behaves as  $[\ln(x) - 1] x^{\eta-1}$ . Thus, with  $\eta > 0$ , no singularity arises<sup>41</sup> from the integration at  $x = 0$ . This result completes the conditions necessary (in the second proof) to show that  $\sigma(p^2)$  goes to zero at large momenta and that the IR singularity  $-\mathcal{D}(0) \ln(p^2)$  appears in the limit  $p^2 \rightarrow 0$ .

Finally, due to the well-known results

$$\lim_{x \rightarrow 0} x^\epsilon \ln^a(x) = \lim_{x \rightarrow \infty} \frac{\ln^a(x)}{x^\epsilon} = 0 \quad (\text{C16})$$

for  $\epsilon > 0$ , it is clear that the above proofs can also be generalized to asymptotic behaviors that include logarithmic functions. At large momenta these logarithmic corrections could be present, for example, if one uses, as an input in the evaluation of  $\sigma(p^2)$ , a gluon propagator obtained in perturbation theory beyond the tree-level term.

#### Appendix D: The $d = 4$ Case Using a MOM Scheme

Let us start from Eq. (B32), with  $\nu = 1$  and  $\ell = \infty$ , and subtract  $\sigma(\mu^2)$  from  $\sigma(p^2)$ , where  $\mu$  is a fixed momentum. Then we can write

$$\begin{aligned} \frac{\sigma(p^2) - \sigma(\mu^2)}{g^2 N_c} &= \frac{\Omega_d}{(2\pi)^d} \frac{d-1}{d} \left\{ \int_0^p dq q^{d-1} \mathcal{D}(q^2) \frac{{}_2F_1(1, 1-d/2; 1+d/2; q^2/p^2)}{p^2} \right. \\ &\quad - \int_0^\mu dq q^{d-1} \mathcal{D}(q^2) \frac{{}_2F_1(1, 1-d/2; 1+d/2; q^2/\mu^2)}{\mu^2} \\ &\quad + \int_p^\mu dq q^{d-3} \mathcal{D}(q^2) {}_2F_1(1, 1-d/2; 1+d/2; p^2/q^2) \\ &\quad + \int_\mu^\infty dq q^{d-3} \mathcal{D}(q^2) [{}_2F_1(1, 1-d/2; 1+d/2; p^2/q^2) \\ &\quad \left. - {}_2F_1(1, 1-d/2; 1+d/2; \mu^2/q^2)] \right\}. \quad (\text{D1}) \end{aligned}$$

<sup>39</sup> As before, one needs to be careful and make hypotheses on  $\mathcal{D}'(p^2)$  and not on  $\mathcal{D}(p^2)$ . For example, after making assumptions on the UV behavior of  $\mathcal{D}(p^2)$  one could employ de l'Hôpital's rule in order to obtain

$$0 = \lim_{p^2 \rightarrow \infty} \mathcal{D}(p^2) \ln(p^2) = \lim_{p^2 \rightarrow \infty} \frac{\mathcal{D}(p^2)}{\frac{1}{\ln(p^2)}} = \lim_{p^2 \rightarrow \infty} \frac{\mathcal{D}'(p^2)}{-\frac{1}{p^2 \ln^2(p^2)}} = - \lim_{p^2 \rightarrow \infty} p^2 \ln^2(p^2) \mathcal{D}'(p^2) \quad (\text{C12})$$

and conclude that  $\lim_{p^2 \rightarrow \infty} [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2) = 0$  follows from the condition  $\lim_{p^2 \rightarrow \infty} \ln(p^2) \mathcal{D}(p^2) = 0$ . However, it is easy to find counterexamples to this result. A classical one is  $\mathcal{D}(p^2) = \sin(p^2)/p^2$  for which  $\lim_{p^2 \rightarrow \infty} [p^2 \ln(p^2) - p^2] \mathcal{D}'(p^2)$  is undetermined even though  $\lim_{p^2 \rightarrow \infty} \ln(p^2) \mathcal{D}(p^2) = 0$  clearly holds. Thus, in order to apply the result in Eq. (C12), additional auxiliary (also called Tauberian) conditions have to be imposed to the function  $\mathcal{D}(p^2)$ , for example on  $\mathcal{D}'(x)$  or on  $\mathcal{D}''(x)$  (see again Ref. [106]). In particular, imposing  $\mathcal{D}(p^2) > 0$  is not sufficient since  $\mathcal{D}(p^2) = [2 + \sin(p^2)]/p^2$  is also a counterexample. Here we find it simpler to make hypotheses directly on the asymptotic behavior of  $\mathcal{D}'(p^2)$ . However, one should also notice that the large- $x$  behavior  $\mathcal{D}'(p^2) \sim 1/p^{2+2\epsilon}$ , with  $1 > \epsilon > 0$ , could also imply  $\mathcal{D}(p^2) \sim 1/p^{2\epsilon} + \text{constant}$  (see for example Ref. [105]). Thus, we also have to impose explicitly the condition  $\mathcal{D}(p^2) \rightarrow 0$  for  $p^2 \rightarrow \infty$ .

<sup>40</sup> Again, instead of integrating by parts, we could also make hypotheses on the second derivative  $\mathcal{D}''(x)$ .

<sup>41</sup> For the first term this can be easily shown by integrating by parts.

In the case  $d = 4$  we can use the result (B9) in Appendix B. Thus, the last integral in the above expression becomes

$$\frac{\mu^2 - p^2}{3} \int_{\mu}^{\infty} dq \frac{\mathcal{D}(q^2)}{q}, \quad (\text{D2})$$

which is UV-finite for any gluon propagator that goes to zero at large momenta. The apparent linear divergence for large  $p^2$  in the above integral is of course canceled by the third integral in Eq. (D1) above, i.e. by

$$\int_p^{\mu} dq q \mathcal{D}(q^2) \left(1 - \frac{p^2}{3q^2}\right) = \int_{\mu}^p dq \frac{\mathcal{D}(q^2)}{3q} (p^2 - q^2). \quad (\text{D3})$$

Then, for  $p^2 \rightarrow \infty$  and for  $\mathcal{D}(q^2) \sim 1/q^2$  at large momenta one only gets a logarithmic contribution  $-\ln(p)$ , as expected.

Using the above formula (D1), the proof that  $\sigma(p^2)$  is IR-finite for  $\mathcal{D}(p^2) > 0$  can be obtained as in Section II D. Indeed, for  $p^2$  going to zero we have to consider only the first and the third integrals<sup>42</sup> in Eq. (D1). Then, using again the result (B9) we can write

$$\int_0^p dq q^3 \frac{\mathcal{D}(q^2)}{p^2} \left(1 - \frac{q^2}{3p^2}\right) + \int_p^{\mu} dq q \mathcal{D}(q^2) \left(1 - \frac{p^2}{3q^2}\right) \leq \int_0^p dq q^3 \frac{\mathcal{D}(q^2)}{p^2} + \int_p^{\mu} dq q \mathcal{D}(q^2) \quad (\text{D4})$$

and no singularity arises in the limit  $p^2 \rightarrow 0$  if  $\mathcal{D}(0) > 0$ . At the same result one arrives by setting  $\mathcal{D}(q^2) = \mathcal{D}(0)$  and by integrating explicitly the l.h.s. in the above equation.

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- [1] R. Alkofer, L. von Smekal, Phys. Rept. **353**, 281 (2001).
  - [2] P. Maris, C. D. Roberts, Int. J. Mod. Phys. **E12**, 297 (2003).
  - [3] A. A. Natale, Braz. J. Phys. **37**, 306 (2007).
  - [4] D. Dudal, M. S. Guimaraes, S. P. Sorella, Phys. Rev. Lett. **106**, 062003 (2011).
  - [5] J. Greensite, Lect. Notes Phys. **821**, 1 (2011).
  - [6] D. J. Gross, R. D. Pisarski, L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
  - [7] U. Kraemmer, A. Rebhan, Rept. Prog. Phys. **67**, 351 (2004).
  - [8] T. Schafer, [hep-ph/0509068].
  - [9] O. Piguet, S. P. Sorella, Lect. Notes Phys. **M28**, 1 (1995).
  - [10] A. G. Williams, “Dyson-Schwinger equations in nonperturbative field theory,” ADP-94-8-T-150.
  - [11] E. S. Swanson, AIP Conf. Proc. **1296**, 75 (2010).
  - [12] P. Boucaud *et al.*, [arXiv:1109.1936 [hep-ph]].
  - [13] L. von Smekal, A. Hauck, R. Alkofer, Annals Phys. **267**, 1 (1998).
  - [14] D. Zwanziger, Phys. Rev. **D65**, 094039 (2002).
  - [15] C. Lerche, L. von Smekal, Phys. Rev. **D65**, 125006 (2002).
  - [16] M. Q. Huber, R. Alkofer, C. S. Fischer, K. Schwenzer, Phys. Lett. **B659**, 434 (2008).
  - [17] R. Alkofer, J. Greensite, J. Phys. **G34**, S3 (2007).
  - [18] A. C. Aguilar, A. A. Natale, JHEP **0408**, 057 (2004).
  - [19] P. Boucaud *et al.*, JHEP **0606**, 001 (2006).
  - [20] A. C. Aguilar, D. Binosi, J. Papavassiliou, Phys. Rev. **D78**, 025010 (2008).
  - [21] P. Boucaud *et al.*, JHEP **0806**, 012 (2008).
  - [22] P. Boucaud *et al.*, JHEP **0806**, 099 (2008).
  - [23] D. Binosi, J. Papavassiliou, Phys. Rept. **479**, 1 (2009).
  - [24] A. C. Aguilar, D. Binosi, J. Papavassiliou, Phys. Rev. **D81**, 125025 (2010).
  - [25] M. R. Pennington, D. J. Wilson, Phys. Rev. **D84**, 119901 (2011).

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<sup>42</sup> Clearly, the second integral does not depend on  $p^2$  and the last one is regular at  $p^2 = 0$  [see Eq. (D2)].

- [26] A. C. Aguilar, D. Ibanez, V. Mathieu, J. Papavassiliou, Phys. Rev. **D85**, 014018 (2012).
- [27] J. M. Cornwall, Phys. Rev. **D26**, 1453 (1982).
- [28] E. R. Caianiello, G. Scarpetta, Nuovo Cim. **A22**, 448 (1974).
- [29] C. M. Bender, F. Cooper, L. M. Simmons, Phys. Rev. **D39**, 2343 (1989).
- [30] “Path Integral Methods in Quantum Field Theory”, R. J. Rivers, (Cambridge University Press, 1987).
- [31] C. S. Fischer, A. Maas, J. M. Pawłowski, Annals Phys. **324**, 2408 (2009).
- [32] J. C. R. Bloch, Few Body Syst. **33**, 111 (2003).
- [33] A. Weber, arXiv:1112.1157 [hep-th].
- [34] J. Rodriguez-Quintero, AIP Conf. Proc. **1354**, 118 (2011).
- [35] V. N. Gribov, Nucl. Phys. **B139**, 1 (1978).
- [36] N. Vandersickel and D. Zwanziger, arXiv:1202.1491 [hep-th].
- [37] I. M. Singer, Commun. Math. Phys. **60**, 7 (1978).
- [38] T. P. Killingback, Phys. Lett. **B138**, 87 (1984).
- [39] P. Hirschfeld, Nucl. Phys. **B157**, 37 (1979).
- [40] D. Zwanziger, Nucl. Phys. **B192**, 259 (1981).
- [41] C. Parrinello, G. Jona-Lasinio, Phys. Lett. **B251**, 175 (1990).
- [42] D. Zwanziger, Nucl. Phys. **B412**, 657 (1994).
- [43] G. Dell’Antonio, D. Zwanziger, Commun. Math. Phys. **138**, 291 (1991).
- [44] R. Friedberg, T. D. Lee, Y. Pang, H. C. Ren, Annals Phys. **246**, 381 (1996).
- [45] L. Baulieu, D. Zwanziger, JHEP **0108**, 016 (2001).
- [46] D. Zwanziger, Nucl. Phys. **B209**, 336 (1982).
- [47] G. Dell’Antonio, D. Zwanziger, Nucl. Phys. **B326**, 333 (1989).
- [48] P. van Baal, Nucl. Phys. **B369**, 259 (1992).
- [49] E. Marinari, C. Parrinello, R. Ricci, Nucl. Phys. **B362**, 487 (1991).
- [50] G. Parisi, arXiv:cond-mat/9411115.
- [51] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, A. Sternbeck, PoS **LAT2007**, 290 (2007).
- [52] A. Cucchieri, T. Mendes, PoS **LAT2007**, 297 (2007).
- [53] A. Sternbeck, L. von Smekal, D. B. Leinweber, A. G. Williams, PoS **LAT2007**, 340 (2007).
- [54] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, A. Sternbeck, Phys. Lett. **B676**, 69 (2009).
- [55] A. Cucchieri, T. Mendes, PoS **QCD-TNT09**, 026 (2009).
- [56] A. Maas, Phys. Rev. **D75**, 116004 (2007).
- [57] A. Cucchieri, T. Mendes, Phys. Rev. Lett. **100**, 241601 (2008).
- [58] A. Cucchieri, T. Mendes, AIP Conf. Proc. **1343**, 185 (2011).
- [59] A. Cucchieri, D. Dudal, T. Mendes, N. Vandersickel, [arXiv:1111.2327 [hep-lat]].
- [60] A. Cucchieri, D. Dudal, T. Mendes, N. Vandersickel, PoS **QCD-TNT-II**, 030 (2011).
- [61] A. Cucchieri, Nucl. Phys. B **508**, 353 (1997).
- [62] P. J. Silva, O. Oliveira, Nucl. Phys. B **690**, 177 (2004).
- [63] I. L. Bogolubsky, G. Burgio, M. Muller-Preussker, V. K. Mitrjushkin, Phys. Rev. D **74**, 034503 (2006).
- [64] V. G. Bornyakov, V. K. Mitrjushkin, M. Muller-Preussker, Phys. Rev. D **79**, 074504 (2009).
- [65] A. Maas, Phys. Rev. D **79**, 014505 (2009).
- [66] V. G. Bornyakov, V. K. Mitrjushkin, R. N. Rogalyov, [arXiv:1112.4975 [hep-lat]].
- [67] A. Maas, Phys. Lett. **B689**, 107 (2010).
- [68] A. Maas, J. M. Pawłowski, D. Spielmann, A. Sternbeck, L. von Smekal, Eur. Phys. J. **C68**, 183 (2010).
- [69] A. Maas, PoS **LATTICE2010**, 279 (2010).
- [70] A. Maas, PoS **QCD-TNT-II**, 028 (2011).
- [71] D. Zwanziger, Nucl. Phys. **B323**, 513 (1989).
- [72] D. Zwanziger, Nucl. Phys. **B399**, 477 (1993).
- [73] N. Maggiore, M. Schaden, Phys. Rev. **D50**, 6616 (1994).
- [74] D. Dudal, S. P. Sorella, N. Vandersickel, Eur. Phys. J. **C68**, 283 (2010).
- [75] D. Zwanziger, AIP Conf. Proc. **1343**, 176 (2011).
- [76] D. Zwanziger, PoS **FACESQCD**, 023 (2010).
- [77] J. A. Gracey, Phys. Lett. B **632**, 282 (2006) [Erratum-ibid. **686**, 319 (2010)].
- [78] J. A. Gracey, JHEP **0605**, 052 (2006).

- [79] J. A. Gracey, Braz. J. Phys. **37**, 226 (2007).
- [80] F. R. Ford, J. A. Gracey, J. Phys. **A42**, 325402 (2009).
- [81] J. A. Gracey, Eur. Phys. J. **C70**, 451 (2010).
- [82] D. Dudal, J. A. Gracey, S. P. Sorella, N. Vandersickel, H. Verschelde, Phys. Rev. **D78**, 065047 (2008).
- [83] D. Dudal, J. A. Gracey, S. P. Sorella, N. Vandersickel, H. Verschelde, Phys. Rev. **D78**, 125012 (2008).
- [84] N. Vandersickel, [arXiv:1104.1315 [hep-th]].
- [85] D. Dudal, S. P. Sorella, N. Vandersickel, Phys. Rev. **D84**, 065039 (2011).
- [86] M. Frasca, Phys. Lett. **B670**, 73 (2008).
- [87] K. -I. Kondo, Phys. Lett. **B678**, 322 (2009).
- [88] D. Dudal, O. Oliveira, N. Vandersickel, Phys. Rev. **D81**, 074505 (2010).
- [89] D. Dudal, S. P. Sorella, N. Vandersickel, H. Verschelde, Phys. Lett. **B680**, 377 (2009).
- [90] R. F. Sobreiro, S. P. Sorella, [hep-th/0504095].
- [91] A. J. Gomez, M. S. Guimaraes, R. F. Sobreiro, S. P. Sorella, Phys. Lett. **B683**, 217 (2010).
- [92] M. Tissier, N. Wschebor, Phys. Rev. **D82**, 101701 (2010).
- [93] M. Tissier, N. Wschebor, Phys. Rev. **D84**, 045018 (2011).
- [94] D. Atkinson, J. C. R. Bloch, Phys. Rev. **D58**, 094036 (1998).
- [95] A. Cucchieri, D. Dudal, T. Mendes, N. Vandersickel, in preparation.
- [96] <http://www.mathpropress.com/stan/bibliography/complexSquareRoot.pdf>
- [97] “Table of Integrals, Series, and Products”, I. S. Gradshteyn and I. M. Ryzhik, Edited by Alan Jeffrey and Daniel Zwillinger (Academic Press, 2007), 7th ed.
- [98] A. Cucchieri, T. Mendes, A. Mihara, JHEP **0412**, 012 (2004).
- [99] A. Cucchieri, A. Maas, T. Mendes, Phys. Rev. **D74**, 014503 (2006).
- [100] E. -M. Ilgenfritz, M. Muller-Preussker, A. Sternbeck, A. Schiller, I. L. Bogolubsky, Braz. J. Phys. **37**, 193 (2007).
- [101] A. Cucchieri, A. Maas, T. Mendes, Phys. Rev. **D77**, 094510 (2008).
- [102] D. Dudal, M. S. Guimaraes, Phys. Rev. **D83**, 045013 (2011).
- [103] J. C. Taylor, Nucl. Phys. **B33**, 436 (1971).
- [104] [http://www.famaf.unc.edu.ar/publicaciones/documents/serie\\_b/BMat48-2.pdf](http://www.famaf.unc.edu.ar/publicaciones/documents/serie_b/BMat48-2.pdf)
- [105] <http://eaton.math.rpi.edu/faculty/Kapila/COURSES/S05/CVITA/ch2.pdf>
- [106] “Asymptotic Methods in Analysis”, N. G. De Bruijn (North-Holland Publishing Co., 1958).